MENSURATION.

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EXPLANATORY

MENSURATION

FOR

THE USE OF SCHOOLS.

CONTAINING NUMEROUS EXAMPLES, AND (BY THE KIND PERMISSION OF THE OXFORD DELEGATES) EMBODYING NEARLY ALL THE QUESTIONS SET IN THEIR LOCAL EXAMINATION PAPERS.

BY THE

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PREFACE.

So NUMEROUS are the purposes to which Mensuration is applied, that it would be almost impossible to exaggerate its importance.

To many boys also, whose time for instruction is likely to be limited, or who, perhaps, would find Euclid too difficult for them, it is believed that Mensuration would form an excellent substitute.

But, important as Mensuration doubtless is, still, in the opinion of many Masters, the works that have recently appeared on this subject are *too long* to be used as ordinary School Books. Acting on this impression, the Author has endeavoured to supply a work on Mensuration which a boy of ordinary ability could hope to get through within a *reasonable* time.

The Questions, which number about 700, are, for the most part, original; but, by the kind permission of the Oxford Delegates for Conducting the Local Examinations, nearly all the questions that have appeared in their Examination Papers have been embodied in the present Work.

This feature of the Work will doubtless prove most useful, as it will show candidates for these Examinations what is the *style* of question that they may expect—those that have appeared in the Junior Examination

Papers being marked by a single asterisk (*), whilst those that have been set to the Senior Candidates have prefixed to them a double asterisk (**).

Whilst framing the Questions with the express intention of impressing the different rules upon the Student's mind, care has been taken to avoid wearying him with long and tedious *Arithmetical* calculations.

But *before* commencing Mensuration, it is desirable that the Pupil should be well grounded in his Arithmetic, so that his onward course may not be impeded by his ignorance of any particular rule that he may require.

The explanations have been made as clear and as simple as possible, and will be sufficient, it is hoped, to enable the Pupil to work nearly all the Questions without much help from his Master.

Of course, in a work of this kind, containing so many questions, there will necessarily be some mistakes, but it is hoped that they will be found very few indeed, as the greatest care has been taken to make it as correct as possible.

Any corrections, however, or suggestions for its improvement, will be most thankfully received by the Author.

A. HILEY.

THORP-ARCH SCHOOL, YORKSHIRE:
June 20, 1871.

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MENSURATION.

DEFINITIONS.

I. Mensuration enables us to find the length of lines, the area of surfaces, and the volume of solids.

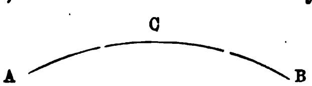
II. A point in Geometry is considered as having neither length, nor breadth, nor thickness.

III. A line has length, but is considered as having neither breadth nor thickness.

IV. Lines may be either straight, curved, or parallel.

V. A straight line, such as AB, lies evenly between its extreme points A and B; or it may be defined to be, the shortest distance between its extremities.

VI. A curved line, as ACB, is one that is continually changing its direction; or is a line in which no part of it is straight.



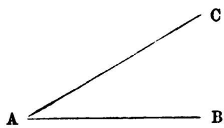
VII. Parallel lines are those which always remain the same distance from each other, however far they may be produced.

VIII. A superficies or surface has length and breadth only; and it is called a *plane* superficies if it is perfectly even or level—such as the top of a table, or a well-laid floor. There are surfaces which are not even or level, such as the curved surface of a globe.

The word *plane* in this definition, and in all other cases where it occurs in this book, means simply *even* or *level*.

IX. A superficies or surface may be contained within *one curved* line, as in the case of a circle; but it cannot be contained within fewer than *three straight* lines.

X. A plane rectilineal angle is the inclination of two straight lines to one another which meet together, but are not in the same straight line.

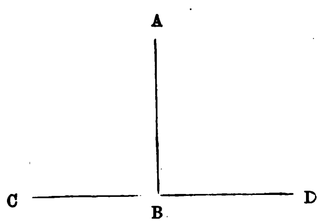


Thus the two straight lines CA and BA, meeting together at the point A, make the angle BAC, or, as it may be called simply, the angle at A.

If an angle is expressed by three letters, as BAC or CAB, the letter A, which stands at the angular point, is always the *middle* letter.

Also it is important to remember that the magnitude of an angle depends *not* upon the *length* of the sides AB, AC, but upon the *extent* or *opening* between the lines.

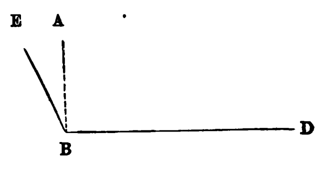
XI. When a straight line standing upon another straight line makes the adjacent angles equal to one another, each of them is called a right angle; and the straight line standing upon the other is called a perpendicular to it.



Thus if AB, standing upon CD, make the angle ABD equal to the angle ABC, then each of the angles ABC or ABD is a right angle; and also the line AB is *perpendicular* to CD.

A right angle contains 90°.

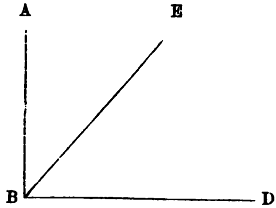
XII. An obtuse angle is greater than a right angle.



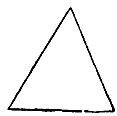
Thus the angle DBE is an obtuse angle, being greater than the right angle DBA.

XIII. An acute angle is an angle that is less than a right angle.

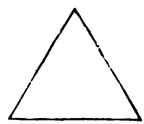
Thus DBE is an acute angle, being less than the right angle DBA.



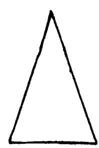
XIV. A triangle is a plane figure contained by three straight lines.



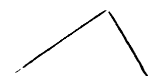
XV. An equilateral triangle has three equal sides. The angles also of an equilateral triangle are all equal.



XVI. An isosceles triangle has two equal sides.



XVII. A scalene triangle has three unequal sides.

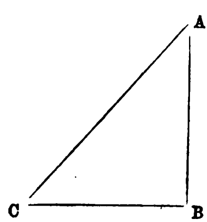


XVIII. A right-angled triangle is one that contains a right angle.

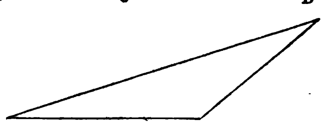
Thus ABC is a right-angled triangle, having the right angle ABC.

AC is called the hypothenuse; AB, the perpendicular; BC, the base.

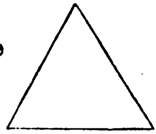
AB and BC are sometimes called the *sides* of a right-angled triangle.



XIX. An obtuse-angled triangle is one that contains an obtuse angle.

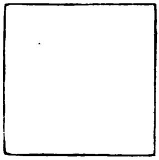


XX. An acute-angled triangle is one which has three acute angles.

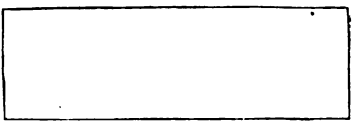


XXI. A quadrilateral is a plane figure bounded by four straight lines; and when its opposite sides are equal and parallel, it is called a parallelogram.

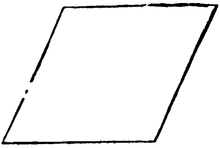
XXII. A square is a four-sided figure having all its sides equal, and all its angles right angles.



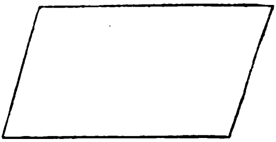
XXIII. A rectangle, oblong, or rectangular parallelogram is a four-sided figure having its opposite sides equal and parallel, and all its angles right angles. The length of a rectangle exceeds its breadth.



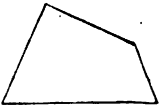
XXIV. A rhombus is a four-sided figure having all its sides equal, but its angles are not right angles.

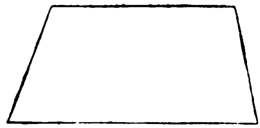


XXV. A rhomboid is a four-sided figure having its opposite sides equal and parallel, but its angles are not right angles.



XXVI. A trapezium is a four-sided figure having no parallel sides.





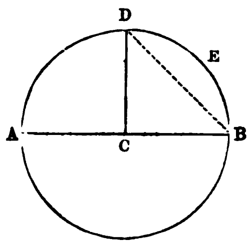
XXVII. A trapezoid is a four-sided figure having only two of its sides parallel.

XXVIII. A polygon is a figure contained by three or more sides, though it is *generally* the name given to a figure bounded by more than *four* straight lines.

The following are the names of the different polygons :—

The Pentagon	has	5	sides	The Nonagon	has	9	sides
„ Hexagon	„	6	„	„ Decagon	„	10	„
„ Heptagon	„	7	„	„ Undecagon	„	11	„
„ Octagon	„	8	„	„ Dodecagon	„	12	„

If all the sides of a polygon are equal, and all its angles equal, then it is called a regular polygon; but if the sides are unequal, and also the angles unequal, then it is an irregular polygon.



XXIX. A circle is a plane figure contained by one line, called the circumference, and is such that all the straight lines drawn from a certain point within it to the circumference are equal to one another.

This point (C) is called the centre of the circle.

XXX. A radius is a straight line, as CA, CD, or CB, drawn from the centre C to the circumference.

XXXI. A diameter is a straight line (AB) passing through the centre and terminated both ways by the circumference.

XXXII. A semicircle is half a circle, or is the figure contained by a diameter and part of the circumference (ADB) cut off by the diameter, as ADB.

XXXIII. A quadrant is the fourth part of a circle, as BCD or DCA.

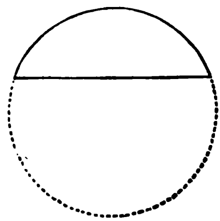
XXXIV. The circumference of any circle is divided into 360 parts, called degrees.

XXXV. An arc is any part of the circumference, as BED.

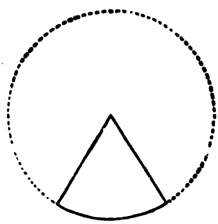
XXXVI. The chord of an arc is a straight line which joins the ends of an arc. Thus BD is a chord joining B and D, the extremities of the arc BED.

Thus the straight line BD is the chord of the arc BED.

XXXVII. A segment of a circle is any part of a circle bounded by a chord and the arc which it cuts off.



XXXVIII. A sector of a circle is any part of a circle bounded by two radii and the arc between them.



TABLES.

Lineal Measure.

12 inches = 1 foot.
3 feet = 1 yard.
$5\frac{1}{2}$ yards = 1 pole.
40 poles or 220 yards = 1 fur.
8 furlongs or 1,760 yds. = 1 mile.

Also $7\frac{23}{25}$ inches = 1 link.
100 links or 22 yds. = 1 chain.
80 chains = 1 mile.

Square Measure.

144 sq. inches = 1 sq. foot.
9 sq. feet = 1 sq. yard.
$30\frac{1}{4}$ sq. yards = 1 sq. pole.
40 sq. poles = 1 rood.
4 roods or 4,840 sq. yds. = 1 acre.

Also 625 sq. links = 1 sq. pole.
10,000 sq. links = 1 sq. chain (484 sq. yds.).
10 sq. chains or 100,000 sq. links = 1 acre.

To the Student.

[All the Examples that have an asterisk (*) prefixed to them have appeared in the Papers of the Junior Candidates at the Oxford Local Examinations; whilst the Questions that have been given to the Senior Candidates are marked with a double asterisk (**).]

I. THE SIDES OF A RIGHT-ANGLED TRIANGLE.

Definitions.—A right-angled triangle is one that contains a right angle.

A C is the *hypothenuse.*

B C is the *base.*

A B, the *perpendicular.*

A B and B C, which include the right angle, are called the *sides* of the right-angled triangle.

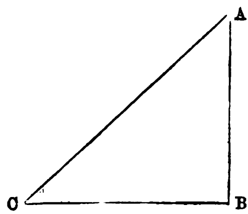


Fig. 1.

RULES.—(1) *To find the hypothenuse, when the base and the perpendicular are given.*

Square the base, and square the perpendicular; add them together; the square root of the sum is the hypothenuse.

(2) *To find either the base or the perpendicular, when the hypothenuse and the other side are given.*

Square the hypothenuse, and square the side given; subtract the latter from the former: the square root of the difference is the required side.

Or, multiply the sum of the hypothenuse and the side given by their difference; the square root of the product is the required side.

Note 1.—In Euclid (1–47), it has been proved that if a square be described upon AC, the hypothenuse of a right-angled triangle ABC; and if squares also be described upon AB and BC, the other sides of the triangle; then the square upon AC will be equal to the sum of the squares on AB and BC. The rules given above are derived from this proposition.

Note 2.—The three angles of any triangle are equal to two right angles, or 180°; hence, if the angle at B is a right angle, and the angle at C is 45°, then the angle at A is 45°, and therefore the sides AB and BC, opposite these equal angles, are equal—that is, AB=BC.

Note 3.—If the three angles of a triangle are all equal, that is, each angle equal to 60°, then all the three sides of the triangle will be equal to each other.

Note 4.—If the angle at C, in a right-angled triangle, is 30°, then the angle at A will be 60°. Doubling the triangle on the other side of BC, we shall have an equiangular triangle ACD, which will therefore have all its sides equal; hence AC=CD=AD, and therefore AB=$\frac{1}{2}$AC.

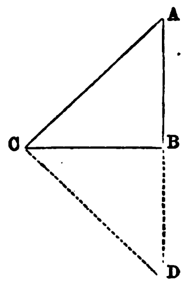


Fig. 2.

Note 5.—Supposing the angle at C, in a right-angled triangle ABC, is 60°; then the angle at A is 30°. Hence, doubling the triangle on the other side of AB, we have the equiangular, and therefore equilateral, triangle ACD—that is, AC=CD=AD. Therefore CB=$\frac{1}{2}$CD=$\frac{1}{2}$AC. A thorough understanding of these remarks will enable the student, without the use of Trigonometry, to find the

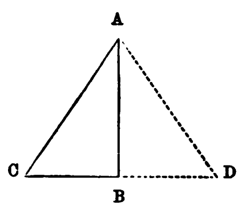


Fig. 3.

remaining sides of a right-angled triangle, when only one side is given, and the angle at C is either 30°, 45°, or 60°.

Note 6.—When questions occur respecting the gable-end of a house, the following explanation of terms must be borne in mind.

CD or EF is the breadth of the house, or width, or span of the gable.

C and D, the points of junction of roof with the wall, are called the eaves.

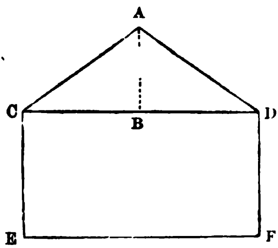


Fig. 4.

A is the ridge or the highest point of the roof.

AB is the perpendicular height of the roof.

The whole figure ACEFD is called the gable-end of the house; and the triangle ACD is called the gable-top.

Note 7.—The diagonal AC of a square or rectangle is the hypothenuse of the right-angled triangle ABC, or of the right-angled triangle ADC.

The length and breadth of the square or rectangle (which in the case of the square are equal to each other) are the sides of the right-angled triangle.

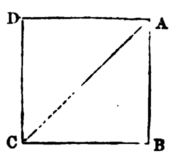


Fig. 5.

Example 1.—The base of a right-angled triangle is 63, and its perpendicular is 16; find its hypothenuse.

The square of 63 is 3,969; and the square of 16 is 256. Adding these together, we have 4,225. Then take the square root of 4,225, which is 65, the length of the hypothenuse.

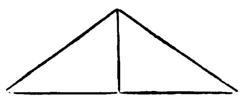
Example 2.—If the perpendicular is 12 ft. 9 in. and the hypothenuse is 15 ft. 5 in., find the length of the base.

Reduce, in the first place, 12 ft. 9 in. and 15 ft. 5 in. to inches, and we shall have 153 inches and 185 inches.

Then the square of 185 is 34,225, and the square of 153 is 23,409; and the difference between them is 10,816.

Taking the square root of 10,816, we shall have 104 inches, or 8 ft. 8 in.—the base.

Example 3.—The sides AC and AD of the triangle ACD are 55 ft. and 65 ft. respectively, and the perpendicular height AB is 33 ft.; find the base CD.



Now, $CD = CB + BD$;
and $BD = \sqrt{AD^2 - AB^2} = \sqrt{65^2 - 33^2}$
$= \sqrt{4225 - 1089} = \sqrt{3136} = 56$ft.;
again, $CB = \sqrt{AC^2 - AB^2} = \sqrt{55^2 - 33^2}$
$= \sqrt{3025 - 1089} = \sqrt{1936} = 44$ ft.;
then, $CD = CB + BD = 56$ ft. $+ 44$ ft. $= 100$ ft.

Example 4.—If the angle at C (see fig. 2) is 30°, and AB is 20 ft., find the length of the base BC.

By *Note* 4.—Double the triangle on the other side of CB, and we have the equiangular triangle ACD, which has therefore all its sides equal.

Hence $AC = AD = 2 \times AB = 2 \times 20$ ft. $= 40$ ft.

Therefore $BC = \sqrt{AC^2 - AB^2} = \sqrt{40^2 - 20^2} = \sqrt{1600 - 400}$ $= \sqrt{1200} = 34{\cdot}641$ ft.

$$\text{Formulæ}\begin{cases}\text{I. } AC = \sqrt{AB^2 + BC^2} \\ \text{II. } BC = \sqrt{AC^2 - AB^2} = \sqrt{(AC + AB)\ (AC - AB)} \\ \text{III. } AB = \sqrt{AC^2 - BC^2} = \sqrt{(AC + BC)\ (AC - BC)}\end{cases}$$

EXAMPLES.

Find the hypothenuse of the right-angled triangle whose base and perpendicular are, respectively—

(1) Perp. 36, and base 77. (2) Perp. 28, and base 45.
(3) Perp. 19, and base 180.

Find the hypothenuse of the right-angled triangle, when the perpendicular and the base are, respectively—

(4) Perp. 11 ft. 8 in, and base 14 ft. 3 in.
(5) Perp. 24 ft. 9 in., and base 25 ft. 4 in.
(6) Perp. 61·6 yds., and base 66·3 yds.

Determine the base of the right-angled triangle, when its hypothenuse and perpendicular are, respectively—

(7) Hyp. 50 ft., and perp. 30 ft.
(8) Hyp. 31 ft. 5 in., and perp. 12 ft. 8 in.
(9) Hyp. 945 yds. 1 ft., and perp. 345 yds. 1 ft.
(10) Hyp. 22 ft. 5 in., and perp. 21 ft. 8 in.

(11) The side of a square is 100 ft.; find its diagonal.

(12) The diagonal of a square is 33 ft. 4 in.; what is the length of each side?

(13) The diagonal of a rectangle is 46 ft. 5 in., and its length is 44 ft. 4 in.; find its breadth.

(14) The length of each side of an equilateral triangle is 120 ft.; find the length of the perpendicular drawn from any angle to its opposite side.

(15) The slant height of a cone is 7 ft. 1 in., and the diameter of its base 2 ft. 2 in.; find its perpendicular height.

(16) The length of a rectangular room is 27 ft. 8 in., and its breadth is 20 ft. 9 in.; find the distance from one corner of the floor to the opposite corner across the room.

(17) A ladder 65 ft. long reaches a window 56 ft. high from the ground; find the distance of the foot of the ladder from the side of the house.

(18) A ladder 53 ft. long reaches a window 45 ft. high; how much nearer to the side of the house must it be moved that it may reach a window 48 ft. high?

(19) Two columns, whose heights are respectively 17 ft. and 50 ft., stand on the same horizontal plane, with their bases 56 ft. apart; find the distance between the tops of these columns.

(20) Find the cost of building a wall round a garden, in the shape of a right-angled triangle, whose hypothenuse is 97 yds. and base 72 yds., at 13*s.* 6*d.* per yard.

(21) Two persons, travelling at the rate of 6 and 8 miles per hour respectively, proceed in directions at right angles to each other; find the distance between them at the end of 6 hours.

(22) Find the length of a beam which shall rest on two walls standing on the same horizontal plane, and which are respectively 4 ft. and 8 ft. 4 in. high, when the distance between them is 13 ft. 9 in.

(23) The mast of a ship is partially broken by the wind. The broken part, which is 35 ft. long, though still adhering to the unbroken part, strikes with its top the deck at the distance of 28 ft. from the foot of the mast; find the entire length of the mast.

(24) A scaffolding pole 40 ft. long, placed at the distance of 21 ft. from the side of a house, is partially broken by the wind, at the height of 5 ft. from the ground, so that its top strikes against the wall; find at what height from the ground it will strike the wall.

(25) There are two upright pillars, in the same horizontal plane, whose heights are respectively 44 ft. and 28 ft. A certain point is taken in that plane between the two pillars, and it is found that the distance of *this point* from the top of the higher pillar is 125 ft., and from the top of the shorter pillar is 53 ft. Find the distance between the tops of these pillars.

(26) A ladder 55 ft. long may be so placed in a street as to reach a window 44 ft. high on one side, and, on being turned round, without changing its position, it will reach another window 33 ft. high on the opposite side of the street; find the breadth of the street.

(27) A ladder 37 ft. long is so placed in a street 36 ft. wide that it will reach a window 35 ft. high, and, upon being turned round, without changing its position, it will reach another window upon the opposite side of the street; find the height of the window.

(28) The dimensions of a rectangular room are 20 ft.

long, 15ft. wide, and 12 ft. high; find the distance from a corner of the floor to the opposite corner of the ceiling across the room.

(29) A foot-passenger, instead of keeping to the road, which runs along the two adjacent sides of a rectangular field, which measures 640 yds. long and 480 yds. broad, goes across the field from one corner to the opposite corner of it; find what distance he will save by doing so.

(30) The breadth of the gable-end of a house is 56 ft., the perpendicular height of the roof is 15 ft., and the distance of the ridge from one of its eaves is 25 ft.; find the distance of the ridge from the other of the eaves.

(31) If AC, the hypothenuse of a right-angled triangle ABC (fig. 1), is 40 ft., and the angle at A is 45°, find the length of the base BC.

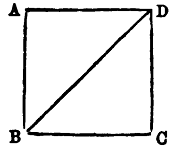
(32) The angle at C, in a right-angled triangle ABC (fig. 2), is 30°, and the perpendicular AB is 30 ft.; find the hypothenuse AC.

(33) BC, the base of a right-angled triangle ABC (fig. 3), is 50 ft., and the angle at C is 60°; find the perpendicular AB.

(34) AC, the hypothenuse of a right-angled triangle ABC (fig. 3), is 80 ft., and the angle at C is 60°; find the length of the base BC, and also of the perpendicular AB.

(35) A mast, being partially broken by the wind, at the height of 20 ft. from its foot, its top touches the deck at an angle of 30°. Find the entire length of the mast.

II. THE SQUARE.



Definition.—A square is a four-sided figure, having all its sides equal, and all its angles right angles.

BD is the diagonal.

The perimeter of a square is the sum of all its sides, or four times the length of a side.

RULES.—(1) *To find the area of a square, when a side is given.*

Square the side given.

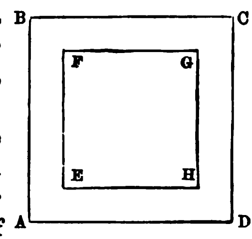
(2) *To find the area of a square, when its diagonal is given.*

Square the diagonal, and divide by 2.

Note 1.—To find the side of a square, when its area is given.—Take the square root of the area.

Note 2.—To find the diagonal of a square, when its area is given.—Take the square root of the double of its area.

Note 3.—To find the number of tiles or flags required for paving a square or rectangular floor or court.—Divide the area of the floor or court by the area of each tile or flag (always bearing in mind that, before dividing, we must have the areas in the same denomination).

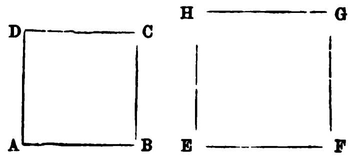


Note 4.—To find the area of a walk or border, surrounding or running round the inside of a square court or plot of land.—First, find the area of the larger square ABCD, including both the court and the walk; and then of the smaller square EFGH. Then, the area of the walk or border is the difference between the areas of these two squares.

N.B.—If EH, the length of the inner square, is given, then AD, the length of the outer square, =(EH + 2 width of walk or border). If AD, the length of the outer square, is given, then EH, the length of the inner square, =(AD − 2 width of walk or border).

Note 5.—To find the side of a square HEFG, whose area shall bear a certain proportion to that of a given square ABCD.—We may use either of the following methods (*a* or *b*) :—

(*a*) Supposing AB, the side of the square ABCD, is given, and it is required to find EF, the side of another square EFGH, whose area is double that of ABCD, we may proceed thus—



Since the square EFGH is double the square ABCD, and the area of ABCD is AB^2 and that of EFGH is EF^2, we have

$$EF^2 = 2 \times AB^2.$$

So, also, the square roots of these quantities are equal; that is—

$$EF = \sqrt{2 \times AB^2} = AB\sqrt{2}.$$

Hence the side EF will be found by *multiplying* AB, the side of the given square, by $\sqrt{2}$.

Again: If the area of the square EFGH is treble that of the square ABCD, then the side EF will be found by *multiplying* AB by $\sqrt{3}$; and so on.

On the other hand, if the area of the required square is half that of the given square, we shall, in that case, find its side by *dividing* the side of the given square by $\sqrt{2}$; and if the area is one-third, then its side is found by *dividing* the side of the given square by $\sqrt{3}$; and so on.

(*b*) The following method will, however, be the easier, perhaps, for beginners :—

If the area of the required square is double that of the given square, then multiply the area of the *given* square by 2, and the square root of this product is the *side* of the *required* square.

If the area is three times as much, then multiply the area of the *given* square by 3; and the square root of this product will be the *side* of the *required* square.

On the other hand, if the area is half as much, then divide the area of the given square by 2; and the square root of this quotient will be the *side* of the *required* square.

The same remarks will apply to the case when the area is one-third, or one-fourth, &c.

$$[\sqrt{2}=1{\cdot}414+; \quad \sqrt{3}=1{\cdot}732+; \quad \sqrt{4}=2, \text{ \&c.}]$$

Example 1.—If the side of a square is 3 ft. 5 in., find its area.

Here 3 ft. 5 in.$=$41 inches; square 41, and we shall have 1681 sq. in., or 11 sq. ft. 97 sq. in., as the area of the square.

Example 2.—How many square tiles, each $4\frac{1}{2}$ inches long, will be required for paving a square court which is 13 ft. 6 in. long?

Now 13 ft. 6 in.$=$162 inches.

Then, area of court$=162^2=26244$ sq. inches.

And area of each tile$=$square of $4\frac{1}{2}=\frac{81}{4}$ sq. inches.

Therefore, number of tiles$=$area of court$\div$area of each tile

$$=26244\div\tfrac{81}{4}=26244\times\tfrac{4}{81}=1296.$$

Example 3.—If the carpeting of a square room with carpet, at 7*s*. 6*d*. per sq. yard, costs £16 17*s*. 6*d*., find the length of the room.

First, if we divide £16 17*s*. 6*d*. by 7*s*. 6*d*., we have 45 sq. yds., or 405 sq. ft., as area of the floor.

Then, taking the square root of 405, we have 20·12+ft.; the length of the room.

$$\text{Formulæ}\begin{cases}\text{I. Area}=\text{side}^2\\ \text{II. Area}=\dfrac{\text{diagonal}^2}{2}\\ \text{III. Side}=\sqrt{\text{area}}\\ \text{IV. Diagonal}=\sqrt{2\,\text{area}}.\end{cases}$$

Examples.

Find the area of the square, when the length of each of its sides is—

(1) 97 in. (2) 2 ft. 7 in. (3) 6·25 ft.

Determine the area of a square field, when the length of each of its sides is—

(4) 160 yds. (5) 3 ch. 25 lks. (6) 1 fur. 20 poles 3 yds.

Find the area of a square field, when its diagonal is—

(7) 162 yds. (8) 7 chains 30 lks. (9) 30 poles 3 yds.

Find the length of each side of a square, when its area is—

(10) 77841 sq. yds. (11) 32 ac. 1 rood 24 sq. poles.

(12) 86 sq. ft. 92·89 sq. in.

(13) What is the length of a square field which contains 15·625 acres?

(14) Find the side of a square field which contains 3 ac. 1 rood 13 poles $5\frac{3}{4}$ sq. yds.

(15*) Find in yards the side of a square field which contains 10 acres.

(16) A square field contains $10\frac{1}{4}$ acres; find the length of each side, and of its diagonal.

(17) The diagonal of a square court is 36 yds.; find its area.

(18) What is the diagonal of a square court, the length of whose side is 35 yds?

(19) A square plot of ground, which is 127 yds. long, has a path 1 yd. wide running round the *inside* of it; what will be the expense of gravelling it, at 6*d.* per square yard?

(20*) Determine the side of a square field which cost £57 15*s.* $2\frac{3}{4}$*d.* trenching, at $2\frac{3}{4}$*d.* per square yd.

(21*) What is the diagonal of a square whose area is 7 sq. inches?

(22) What will be the expense of building a wall round a square plot of ground, which contains 3 acres 1 rood 4 poles 25 sq. yds., at 10*s.* 6*d.* per yd.?

(23) A square room is 15 ft. 6 in. long; what must be the length of a square piece of carpet that shall cover half the room?

(24) A walk 2 ft. 6 in. wide, round the *outside* of a square court, which is 30 yds. long, is to be tiled; find the expense, at 9*d.* per sq. ft.

(25) A path 2 yds. wide surrounds a square field containing 27 ac. 12 poles 1 sq. yd.; determine the area of the field, if the path were ploughed up and enclosed in the field.

(26) The tiling of a square court, at 1*s.* 9*d.* per sq. ft., costs £22 8*s.*; what is the length of the court?

(27) The perimeter of a square field is 684 yds.; find its area.

(28) 2,500 square tiles are required for paving a square courtyard which is 16 ft. 8 in. long; find the length of each tile.

(29) Find the cost of building a wall round a new burial-ground, in the shape of a square, which contains 2 ac. 10 poles 17½ sq. yds., at 17*s.* 6*d.* per yd.

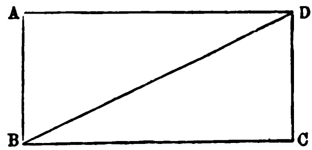
(30) A square field, containing 6 ac. 2 roods 31 poles 2¼ sq. yds., is to be planted with trees, to the depth of 10 yds., running all round the *inside* of the field, at 1*s.* 3*d.* per sq. yd. Find the cost.

(31) A square field 210 yds. long is to be planted all round the *inside* of it to an uniform depth; the plantation is to occupy just one-seventh of the whole field. Find its width.

(32) A square plot of ground is 35 yds. 1 ft. long; what is the side of another plot, which is of the same shape, but which contains our times the amount of land?

III. THE RECTANGLE.

Definitions.—A rectangle is a parallelogram whose angles are all right angles; that is, its adjacent sides are perpendicular to each other.



BC and AD are called *length* or *base*.

AB and DC are called *breadth* or *height*.

BD is the *diagonal.*

The *perimeter* is the sum of all the sides=2 length +2 width.

RULE.—*To find the area of a rectangle.*

Multiply the length by the breadth.

Note 1.—To find either the length or the breadth of a rectangle, when the area and one dimension are given.—The area divided by the breadth will give the length. The area divided by the length will give the breadth.

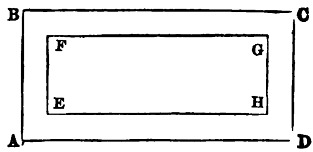
Note 2.—To find the area of a rectangle, when one dimension, either the length or the breadth, and the diagonal, are given.—Since BD is the hypothenuse of a right-angled triangle; and one side also is given, the other side must be found by Rule 2, Prob. 1. We shall then have found the length and breadth of the rectangle; and these, being multiplied together, will give the area of the rectangle.

Note 3.—The *superficial area* of the *walls* of a room, or of the *sides* of a rectangular box or cistern, may be found by adding double the length to double the breadth, and then multiplying this sum by the height of the room, or by the depth of the box or cistern.

The *floor* of a room, or the *bottom* of a box or cistern, will be found by multiplying the length by the breadth.

Note 4.—With regard to finding the number of tiles or planks necessary for tiling or flooring a rectangular room or court, see Note 3, Prob. 2.

Note 5.—To find the area of a uniform walk or border, *surrounding* a rectangle, or running round the *inside* of it.



Find the area of the *outer* rectangle ABCD, and also of the *inner* rectangle EFGH.

Then, the difference of these two is the area of the walk or border.

Observe that $\mathrm{AD}=\overline{\mathrm{EH}+2\text{ width of walk}}$;

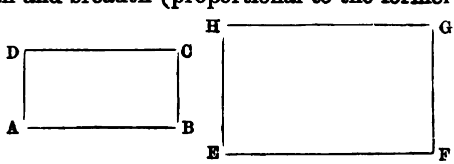
$\mathrm{AB}=\overline{\mathrm{EF}+2\text{ width of walk}}$;

$\mathrm{EH}=\overline{\mathrm{AD}-2\text{ width of walk}}$;

and $\mathrm{EF}=\overline{\mathrm{AB}-2\text{ width of walk}}$.

Note 6.—To ascertain the number of yards of carpet, of a certain width, that will be required for a floor.—Find the area of the floor, and divide it by the width of the carpet. (Observe that the *area* of the floor and the *width* of the carpet must be *both* expressed in the *same* denomination, either yards, feet, or inches; and the *required length* of the carpet will be in the *same* denomination.)

Note 7.—Given the length and breadth of a rectangle, to find the length and breadth (proportional to the former) of another rectangle whose area shall be double, treble, &c. that of the former.



Supposing AB and AD of the rectangle ABCD are given, and it is required to find EF and EH (proportional to AB

and AD) of another rectangle EFGH, whose area shall be twice that of the rectangle ABCD, we have—

$EF \times EH = 2(AB \times AD) = AB\sqrt{2} \times AD\sqrt{2}$ (since $\sqrt{2} \times \sqrt{2} = 2$);

that is, EF, the *length* of *required* rectangle = *length* of *given* rectangle $\times \sqrt{2}$;

and EH, the *breadth* of the *required* rectangle = *breadth* of *given* rectangle $\times \sqrt{2}$.

Hence, if the area is to be doubled, then the length and breadth of the *required* rectangle will be found by multiplying the length and breadth of the *given* rectangle each by $\sqrt{2}$.

If the area is to be trebled, then multiply each of the given dimensions by $\sqrt{3}$; and so on.

Example 1.—Find the area of a rectangular room which is 13 ft. 4 in. long and 12 ft. 8 in. broad.

Area of room = 13 ft. 4 in. × 12 ft. 8 in. = 160 in. × 152 in.
= 24320 sq. in. = 168 sq. ft. 128 sq. in

Example 2.—Find the expense of carpeting a room which measures 18 ft. by 15 ft., with carpet 27 in. wide, worth 5*s*. 4*d*. per yd.

Area of the floor = 18 ft. × 15 ft. = 270 sq. ft. = 30 sq. yds; and 27 in. = $\frac{3}{4}$ yd.

Then, yards of carpet = $30 \div \frac{3}{4} = 30 \times \frac{4}{3} = 40$ yds.

And 40 yds., at 5*s*. 4*d*. per yard, = £10 13*s*. 4*d*.

Example 3.—If it requires 1000 tiles, 8 in. long and $3\frac{1}{2}$ in. wide, for paving the floor of a room which is 14 ft. 7 in. long, find the breadth of the room.

Each tile contains ($8 \times 3\frac{1}{2}$), or 28 sq. inches; and therefore 1000 tiles will contain 28000 sq. in., which is the area of the floor.

Then, if we divide 28000 sq. in. by 14 ft. 7 in.—that is, by 175 in.—we shall have 160 in., or 13 ft. 4 in., the breadth of the floor.

Example 4.—How many square feet are there in a gravel walk, 2 ft. wide, running all round the *outside* of a rectangular grass-plot which is 40 ft. long and 28 ft. broad?

Now the *extreme length* of the rectangle ABCD (see fig. Note 5), which includes the grass plot and the walk$=$ 40 ft.$+$4 ft.$=$44 ft.

And the *extreme breadth*$=$28 ft.$+$4 ft.$=$32 ft.

Then, the area of the *larger* rectangle ABCD$=$44 ft.$\times$ 32 ft.$=$1408 sq. ft.;

And area of *grass-plot* EFGH$=$40ft.$\times$28 ft.$=$1,120 sq. ft.

Therefore the gravel walk$=$1408$-$1120$=$288 sq. ft.

$$\text{Formulæ}\begin{cases}\text{I. Area} = \text{length} \times \text{breadth.}\\[4pt] \text{II. Length} = \dfrac{\text{area}}{\text{breadth}}\\[4pt] \text{III. Breadth} = \dfrac{\text{area}}{\text{length}}\end{cases}$$

EXAMPLES.

Find the area of the rectangle whose length and breadth are, respectively—

(1). Length 37 in. and breadth 27 in.

(2) Length 4 yds. and breadth 3 yds. 2 ft.

(3) Length 10 ft. 7 in. and breadth 9 ft. 5 in.

Find the area of the rectangle whose length and breadth are, respectively—

(4) Length 7 ft. 6 in. and breadth 6 ft. 7 in.

(5) Length 17ft. $10\frac{1}{2}$ in. and breadth $5\frac{1}{2}$ in.

(6) Length 4 yds. 2 ft. 6 in. and breadth 2 yds. 1 ft. 4 in.

Determine the area of the following rectangular fields, whose dimensions are—

(7) 210 yds. by 180 yds.

(8) 5 ch. 20 lks. by 4 ch. 35 lks.

(9) 20 poles 5 yds. by 12 poles 1 yd.

Find the area of the rectangle whose length and diagonal are, respectively—

(10) Diagonal 135 yds. and length 108 yds.

(11) Diagonal 295 yds. and length 236 yds.

(12) Diagonal 8·29 ft. and length 6·29 ft.

Find the breadth of the rectangle when its area and length are, respectively—

(13) Area 155 sq. yds. 5 sq. ft. and length 13 yds. 1 ft.

(14) Area 180 sq. yds. 4 sq. ft. and length 18 yds. 2 ft.

(15) Area 24 sq. yds. 1 sq. ft. 80 sq. in. and length 4 yds. 2 ft. 10 in.

Find the breadth of the rectangular field whose area and length are, respectively—

(16) Area $3\frac{1}{2}$ acres and length 140 yds.

(17) Area 3 ac. 2 roods 10 poles $29\frac{1}{2}$ sq. yds. and length 136 yds.

(18) Area 3 ac. 34 poles and length 12 ch. 85 lks.

(19) Area 5 ac. and length $1\frac{1}{4}$ mile.

(20) The length of a rectangular field is 6 ch. 75 lks. and its breadth 3 ch. 15 lks.; find the rental, at £2 10*s.* per acre.

(21**) A path 8 ft. wide, surrounding a rectangular court 60 ft. long and 36 ft. wide, is to be paved with tiles 9 in. long and 4 in. wide; how many will be required?

(22) A certain street, $\frac{1}{8}$ of a mile long, covers $1\frac{1}{4}$ acre; what is the breadth of the street?

(23) The middle part of a room, which measures 20 ft. 6 in. by 16 ft., is covered with a carpet, which is only 15 ft. 9 in. long and 10 ft. 8 in. wide; how much additional carpet, 27 in. wide, will be required to cover the remaining part of the floor?

(24) What will be the expense of building a wall round a rectangular garden which covers exactly $\frac{3}{4}$ acre, and whose breadth is 30 yds., at 15*s.* 6*d.* per yd.?

(25) What will be the expense of making a footpath, 2 ft.

wide, round the *outside* of a rectangular plot containing 572 sq. yds., and whose length is 26 yds., at 1*s.* 3*d.* per sq. yd.?

(26) A square piece of wood is 2 ft. 6 in. long; what must be the breadth of a rectangular piece 3 ft. 9 in. long, that it may be as large as the square piece?

(27) What will be the expense of paving a hall, 50 yds. long and 50 ft. wide, with marble slabs 1 ft. long and 9 in. broad, the price of the slabs being £5 per dozen.

(28*) A room 39 ft. long requires 36 yds. of carpet, 2 ft. 2 in. wide, to cover it; what is the breadth of the room?

(29) The area of a rectangular field is 5 ac. 1 rood 6 poles $8\frac{1}{2}$ sq. yds., and its length is 256 yds.; find the expense of fencing it, at 1*s.* per yd. Supposing the field were a square and of the same area, what would be the expense in that case?

(30*) A rectangular field contains $3\frac{1}{8}$ acres, and is 100 yds. wide; find its breadth.

(31) A rectangular field is $7\frac{1}{2}$ chains in length, and its area is $5\frac{1}{4}$ acres; what is its breadth?

(32) A rectangle, whose length is four times that of its breadth, and a square have the same perimeter, 100 yds.; which contains the greater area, and by how much?

(33**) A rectangular garden is to be cut off from a rectangular field, so as to contain a quarter of an acre. One side of the field is taken for a side of the plot, and measures $3\frac{1}{2}$ chains; find the length of the other side.

(34) The rental of a rectangular field, whose length is 1 fur. 20 poles, at the rate of £1 13*s.* per acre, is £6 6*s.*; find its breadth.

(35) What will be the expense of paving a courtyard which measures 35 ft. 10 in. by 18 ft. 6 in., at 6*s.* 3*d.* per sq. yd.?

(36) Find the expense of lining the sides and bottom of a rectangular cistern, 12 ft. 9 in. long, 8 ft. 3 in. broad, and

6 ft. 6 in. deep, with lead which costs £1 8*s.* per cwt., and weighs 8 lbs. to the sq. ft.

(37) Find the cost of carpeting a room 18 ft. 9 in. long and 17 ft. 6 in. broad, with carpet 2 ft. wide, at 4*s.* 9*d.* per yd.

(38) The cost of carpeting a room 21 ft. long with carpet 24 in. wide, and worth 4*s.* per yd., is £10 10*s.*; what is the breadth of the room?

(39) The perimeter of a square and also of a rectangle, whose breadth is 14 yds., is 392 yds.; what is the difference in size of these two figures?

(40) What will be the area of a rectangular field whose diagonal is 415 yds. and its breadth 249 yds.?

(41) Find how many yards of paper 27 in. wide will be required for papering a room 18 ft. long, 12 ft. broad, and 11 ft. high.

(42) The flooring of a room 14 ft. 3 in. long and 13 ft. 4 in. broad is composed of planks, each 8 in. wide and 10 ft. long; how many will be required?

(43) What is the cost of papering a room 6 yds. 1 ft. 2 in. long, 6 yds. 0 ft. 4 in. broad, and 12 ft. high, with paper $\frac{1}{6}$ of a yard wide, at $4\frac{1}{2}$*d.* per yd.?

(44) A joiner requires $3\frac{1}{2}$ sq. ft. of wood, and he has only a plank 1 ft. 6 in. wide from which to cut it off; find the length of the piece that he must cut off.

(45) The cost of carpeting a room 15 ft. long with carpet 24 in. wide, worth 4*s.* 6*d.* per yd., is £7 17*s.* 6*d.* What is the breadth of the room?

(46) If the breadth of a rectangular field, which is 75 yds. long, were increased by 10 yds., then its area would be 3 roods 28 poles 23 sq. yds.; find its breadth.

(47) The area of a rectangular field, whose breadth is 119 yds., is 2 ac. 3 roods 32 poles 2 sq. yds. Find what distance a person could save himself by going from one corner to the opposite corner across the field, instead of keeping to the footpath, which runs along the two adjacent sides of the field.

(48) The dimensions of a rectangle are 45 yds. and 28 yds. Will its diagonal be greater or less than the diagonal of a square of the same area?

(49) The building of a wall round a square field, at 4*s.* per yd., costs £112; what would have been the charge if the field had been in the shape of a rectangle, whose length is 196 yds., but still containing the same quantity of land?

(50*) How many square yards of painting are there in a room 20 ft. long, 14 ft. 6 in. broad, and 10 ft. 4 in. high; allowing for a fireplace 4 ft. by 4 ft. 4 in., and 2 windows each 6 ft. by 3 ft. 2 in.?

(51**) Find the expense of covering with lead, at a farthing per sq. in., the inside of a cistern, open at the top, of length 10 ft., width 6 ft., and depth 4 ft.

(52) If 864 planks, each $13\frac{1}{2}$ ft. long, are used in the construction of a platform 54 yds. long and 21 yds. broad; find the width of each plank.

(53) The walls of a room, 21 ft. long, 15 ft. 9 in. wide, and 11 ft. 8 in. high, are painted at an expense of £9 12*s.* 6*d.* Find the additional expense of painting the ceiling, at the same rate.

IV. THE OBLIQUE PARALLELOGRAM, OR THE RHOMBUS AND RHOMBOID.

Definitions.—The rhombus (fig. 1) is a four-sided figure

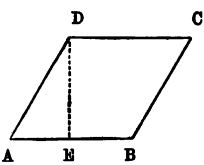


Fig. 1.

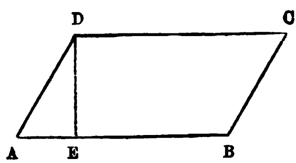


Fig. 2.

which has all its sides equal, but its angles are not right angles.

The rhomboid (fig. 2) is a four-sided figure which has its opposite sides equal and parallel, but all its sides are not equal, and its angles are not right angles.

Since both the rhombus and rhomboid are parallelograms, though their angles are not right angles, the general term *oblique parallelogram* will include both figures.

In both figures, AB is called the *base* or *length*; and DE is the *perpendicular height*, or more generally called simply the *height.*

RULE.—*To find the area of a rhombus or rhomboid.*

Multiply the base (AB) by the perpendicular height (DE).

Note 1.—To find either the base or the perpendicular height, when the area and the other dimension are given.—Divide the area by the base, and the quotient is the perpendicular height; or, divide the area by the perpendicular height, and the quotient is the base.

Note 2.—The reason for the rule given above for finding the area of an oblique parallelogram may be thus shown:—

However oblique a parallelogram ABCD (fig. 3) may be, it has been shown by Euclid (I. 35) that its area is equal to the area of any other parallelogram upon the *same base* AB, and between the *same parallels* AB and CG; and therefore it is equal to the area of the rectangle ABEG.

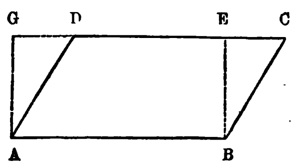


Fig. 3.

But the area of the rectangle ABEG (by Rule, Prob. III.) is$=$AB$\times$BE.

Hence the area of the oblique parallelogram ABCD$=$ AB$\times$BE; that is, $=$base$\times$perpendicular height.

Therefore the area of a parallelogram, whether *rectangular* or *oblique*, will be always found by multiplying the base by

the *perpendicular* let fall upon it from the opposite side. (In the case of a rectangular parallelogram, this perpendicular is the breadth of the figure.)

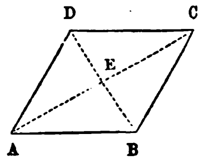
It will be found also that the *more* oblique a parallelogram is, the *less* does its area become, until, at last, AD and BC form but one straight line, when, consequently, the area becomes 0.

Note 3.—If the length AB, and its adjacent side AD, and also AE, the distance intercepted between the angle at A and the perpendicular let fall from D, are given (see figs. 1 and 2), and it is required to find the area of the parallelogram, we must first find the perpendicular DE by Rule 2, Prob. I. (since DE is the perpendicular of a right-angled triangle EAD).

Then, having got the base and the perpendicular height, we can find the area of the oblique parallelogram by the rule given in this chapter.

Note 4.—The diagonals of a rhombus intersect each other at *right angles.* Hence, if the diagonals of a rhombus are given, we may find its area by multiplying together the two diagonals, and dividing the product by 2.

Note 5.—If the diagonals of a rhombus are given, and it is required to find the length of each side of the rhombus, we may proceed thus:—



Since the diagonals bisect each other at right angles, we shall have a right-angled triangle AEB, of which AE is half one diagonal, and BE half the other diagonal. Then AB, the hypothenuse of this right-angled triangle, can be found by Rule 1, Prob. I.

Note 6.—The student must always bear in mind that the area of an *oblique* parallelogram is found *not* by multiplying

together the two adjacent sides AB and AD, as in the case of a rectangular parallelogram, but by multiplying the base AB by DE, the perpendicular let fall upon it from the opposite side (see figs. 1 and 2).

Example 1.—Find the area of a rhombus whose length is 6 chains 25 links, and perpendicular height is 4 chains 50 links.

Now 6 ch. 25 lks. = 625 lks; and 4 ch. 50 lks. = 450 lks.

Then the area = $625 \times 450 = 281250$ sq. links = 2·8125 ac. = 2 ac. 3 roods 10 poles.

Example 2.—The area of a field, in the shape of a rhomboid, is 4 ac. 2 roods 20 poles, and its base is 9 ch. 25 lks.; find its perpendicular height.

Now 4 ac. 2 roods 20 poles = 740 poles = 462500 sq. links; and 9 ch. 25 lks. = 925 links.

Then, perpendicular height = $462500 \div 925 = 500$ links = 5 chains.

$$\text{Formulæ} \begin{cases} \text{I. Area} = \text{base} \times \text{height.} \\ \text{II. Height} = \dfrac{\text{area}}{\text{base.}} \\ \text{III. Base} = \dfrac{\text{area}}{\text{height.}} \end{cases}$$

[The student must bear in mind that when the word *height* occurs in the following Examples, it always refers to the *perpendicular* let fall upon the base from the opposite side.]

Examples.

Find the area of an oblique parallelogram, when its base and perpendicular height, are respectively—

(1) Base 35 in., and height 15 in.

(2) Base 40 ft. 6 in., and height 28 ft. 9 in.

(3) Base 15 yds. 2 ft., and height 10 yds. 1 ft.

Find the area of a field, in the shape of a rhomboid, when its base and height are, respectively—

(4) Base 210 yds., and height 120 yds.

(5) Base 1 fur. 10 poles, and height 25 poles.

(6) Base 6 ch. 25 lks., and height 5 ch. 40 lks.

Find the perpendicular height of an oblique parallelogram, when its area and base are, respectively—

(7) Area 42 sq. ft. 24 sq. in., and base 7 ft. 8 in.

(8) Area 13 ac., and base 440 yds.

(9) Area 4 ac. 1 rood 39 poles, and base 7 ch. 19 lks.

(10) Find the rental of a field, in the shape of a rhomboid, whose base is 13 ch. 75 lks. and height 9 ch. 50 lks., at £3 10*s.* per acre.

(11) The rental of a field, in the shape of a rhomboid, at £2 per acre, is £5 5*s.*, and its base is 6 ch. 25 lks.; find its height.

(12) ABCD (fig. 1) is a rhombus, of which the perimeter is 328 ft., and AE is 18 ft.; find its area.

(13) Find the area of a rhombus whose diagonals are 66 yds. and 120 yds. respectively.

(14) Find the cost of paving a courtyard, in the form of a rhombus, when its diagonals are 45 yds. and 24 yds. respectively, at 3*s.* 4*d.* per sq. yd.

(15) The adjacent sides of a rhomboid (fig. 2) are 64 ft. and 36 ft. If AE = DE, find the area of the figure.

(16) If each side of a rhombus (fig. 1) is 24 ft., find at what distance from A along AB the perpendicular DE must be drawn, so that the area of the rhombus may be $\frac{3}{4}$ the area of a square having the same perimeter.

(17) The diagonals of a rhombus are 90 yds. and 120 yds. respectively; find the length of each of its sides, and also its height.

V. THE TRIANGLE.

Definitions.—A triangle is a plane figure bounded by three straight lines.

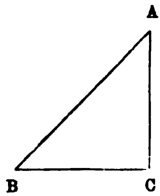


Fig. 1.

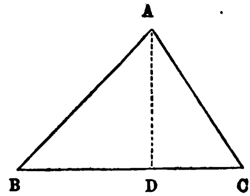


Fig. 2.

A is the *vertex* of the triangle.

BC is the *base*.

AC is the *perpendicular height* in fig. 1.

AD is the *perpendicular height* in fig. 2.

The *perpendicular height* is more frequently called simply the *height*.

Of course, either of the other angular points B or C might be taken as the *vertex*; in which case the side opposite to it would be the *base*, and the perpendicular drawn from that vertex to the opposite side would be the *height* of the triangle.

RULES.—(1) *To find the area of a triangle, when its base and height are given.*

Multiply the base by the height, and divide the product by 2.

(2) *To find the area, when the three sides are given.*

From half the sum of the three sides, subtract each side separately. Multiply the half sum and the three remainders together, and the square root of the product is the area.

Note 1.—To find one dimension, either the base or the height, when the area of the triangle and the other dimer

sion are given.—(*a*) Divide double the area by the base, and the quotient is the height. (*b*) Divide double the area by the height, and the quotient is the base.

Note 2.—The reason for Rule 1 is, that the area of a triangle ACB is half the area of a parallelogram ABCD, upon the *same base* BC, and between the *same parallels* AD and BC (Euclid I.41).

But the area of parallelogram ABCD$=$BC$\times$AB;
therefore area of the triangle ABC $=\frac{1}{2}\times$BC$\times$AB$=\frac{1}{2}\times$base$\times$height.

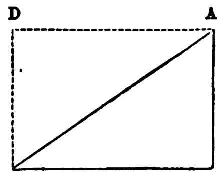


Fig. 3.

Note 3.—The area of triangle ABC $=\frac{1}{2}\times$base BC$\times$perp. height AD.

Because the triangle ABC is equal to the triangle EBC, being upon the same base BC, and between the same parallels EA and CB (Euclid I. 37).

But the triangle EBC$=\frac{1}{2}\times$CB$\times$EB $=\frac{1}{2}\times$CB$\times$AD (since AD$=$EB);
therefore the triangle ABC$=\frac{1}{2}\times$base BC$\times$perp. height AD.

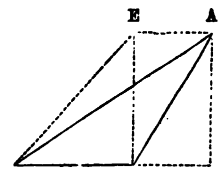


Fig. 4.

Hence, however *oblique* a triangle may be, its area will always be found by multiplying its base CB by AD (which is the perpendicular drawn from the angle at A upon BC produced), and then dividing this product by 2.

Note 4.—To find the length of the perpendicular AD, drawn from the angle at A to the opposite side.

First, find the area of the triangle ABC, either by Rule 1 or Rule 2.

Also, by Rule 1, the area of the triangle ABC is equal to half the product of any side into the perpendicular upon it$=\frac{1}{2}\times$AD.

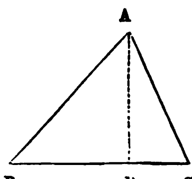


Fig. 5.

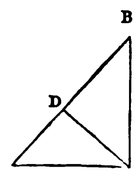


Fig. 6.

And, since both these methods will give the same area, we have—

$$\tfrac{1}{2} \times \text{BC} \times \text{AD} = \text{area of triangle};$$

$$\text{therefore AD} = \frac{2 \text{ area of triangle.}}{\text{BC}}$$

The same method must be adopted in finding the length of the perpendicular upon *any* side.

Note 5.—If it is required to find the area of a right-angled triangle, when one of the other angles and a side are given, we must first find, by Notes 4 and 5, Prob. I., the base and perpendicular height. Having thus found the base and perpendicular height, we can then find the area by Rule 1.

Example 1.—Find the area of a triangle when its base is 12 ft. 6 in., and its perpendicular height is 8 ft. 10 in.

First, 12 ft. 6 in.$=$150 in.; and 8 ft. 10 in.$=$106 in.

Then, area of triangle$=\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 150 \times 106$ $=7950$ sq. in.$=$55 sq. ft. 30 sq. in.

Or thus: 12 ft. 6 in.$=12\frac{1}{2}$ ft.; and 8 ft. 10 in.$=8\frac{5}{6}$ ft.

Then, area of triangle$=\frac{1}{2} \times 12\frac{1}{2} \times 8\frac{5}{6} = \frac{1}{2} \times \frac{25}{2} \times \frac{53}{6} = 55\frac{5}{24}$ sq. ft.$=$55 sq. ft. 30 sq. in.

Example 2.—Find the area of a triangle when its sides are, respectively, 17·5 ft., 22·1 ft., and 31·8 ft.

Now, $17·5+22·1+31·8=71·4$; and half of this is 35·7.

Then $35·7-17·5=18·2$; $35·7-22·1=13·6$; and $35·7-31·8=3·9$.

Then $35·7 \times 18·2 \times 13·6 \times 3·9 = 34462·2096$.

Taking the square root of 34462·2096, we have 185·64 sq. ft.—area of triangle.

Example 3.—The sides of a right-angled triangle are 52 ft. and 39 ft.; find its area, and also the length of the perpendicular drawn from the right angle to the opposite side.

First, the area of the triangle (fig. 6) $=\frac{1}{2}\times$ base AC $\times$ height AB

$$=\tfrac{1}{2}\times 52\times 39=1014 \text{ sq. ft.—1st Answer.}$$

Now, by Rule 1, Prob. I., the hypothenuse BC $=65$ ft.

Then, the area of the triangle also $=\frac{1}{2}\times BC\times AD$

$$=\tfrac{1}{2}\times 65\times AD.$$

And, since this will give the same area (1014 sq. ft.) that we got before, we have—

$$\tfrac{1}{2}\times 65\times AD=1014;$$

therefore $AD=\dfrac{2\times 1014}{65}=\dfrac{2028}{65}=31{\cdot}2$ ft.—2nd Answer.

Formulæ
- I. Area $=\frac{1}{2}\times$ base $\times$ height $=\sqrt{s(s-a)(s-b)(s-c)}$; when $s=\frac{1}{2}$ sum of sides, and a, b, and c the sides respectively.
- II. Base $=\dfrac{2 \text{ area}}{\text{height}}$
- III. Height $=\dfrac{2 \text{ area}}{\text{base}}$.

[Observe, that when the *height* of a triangle is spoken of, we are always to understand its *perpendicular height*; that is, the *perpendicular* drawn from the *vertex* of the triangle to the *base*, or to the *base produced*, if necessary.]

Examples.

Find the area of a triangle, whose base and perpendicular height are, respectively—

(1) Base 50, and perp. ht. 20.

(2) Base 20 ft. 6 in., and perp. ht. 10 ft. 4 in.

(3) Base 14·5 yds., and perp. ht. 10·8 yds.

Determine the area of a triangular field whose base and height are, respectively—

(4) Base 10 ch. 35 lks., and height 8 ch. 32 lks.

(5) Base 1 fur. 5 yds., and height 110 yds.

Find the height of a triangle, when its area and base are, respectively—

(6) Area 102 sq. ft. 72 sq. in., and base 10 ft. 3 in.
(7) Area 108 sq. ft. 68 sq. in., and base 9 ft. 2 in.
(8) Area 5 ac. 0 rood 33 poles, and base 12 ch. 50 lks.
(9) Area 1 ac. 2 roods 17 poles, and base 6 ch. 25 lks.

Find the area of a triangle whose sides are, respectively—

(10) 50, 40, and 30. (11) 61 ft., 91 ft., and 100 ft.
(12) 104 ft., 111 ft., and 175 ft.
(13) 260 yds., 287 yds., and 519 yds.
(14) 11·1 yds., 17·5 yds., and 17·6 yds.
(15) 119 yds., 150 yds., and 241 yds.
(16) 18·6 yds., 22·1 yds., and 27·5 yds.

(17) A plot of land, in the shape of a triangle, whose base is 120 yds. 2 ft., and perpendicular height 55 yds. 1 ft., is to be paved, at the rate of 1*s.* 6*d.* per sq. yd.; find the expense.

(18) How many square feet of brickwork are there in the gable-top of a house, the breadth being 56 ft., and the lengths from the eaves to the ridge 39 ft. and 25 ft. respectively?

(19) The hypothenuse of a right-angled triangle is 185 yds. 2 ft., and the perpendicular height is 55 yds.; find the area.

(20) A plot of land, in the shape of a triangle, whose sides are, respectively, 25 yds., 101 yds., and 114 yds., sells for £1710; find the price per sq. yd.

(21) The perimeter of an equilateral triangle is 60 ft., and it is the same as that of a square; compare the areas of the two figures.

(22) The hypothenuse of a right-angled triangle is 76 yds., and its other sides are equal; find the area of the triangle.

(23) How much land is there in a triangular field whose sides measure, respectively, 2 ch. 73 lks., 4 ch. 25 lks., and 6 ch. 28 lks.?

(24) Find the value of a triangular field, whose sides are, respectively, 113 yds., 225 yds., and 238 yds., at £60 per acre.

(25) The base of a right-angled triangle is 40 ft., and its perpendicular is 30 ft.; find its area. And if a perpendicular be drawn from the right angle upon the hypothenuse, find what is the length of the parts into which the hypothenuse is divided, and the area of each part of the triangle.

(26) Given AC (fig. 2)=24 ft. 7 in., BD=25 ft. 3 in., and CD=14 ft. 9 in.; find the area of the triangle.

(27) The three sides of a triangle (fig. 2) are AC=21 ft., CB=89 ft., and AB=100 ft.; required the length of the perpendicular from C on AB.

(28**) ABC is a triangle, and AD the perpendicular from A upon BC. If AD=13 ft., and the lengths of the perpendiculars from D on AB and AC be 5 ft. and $10\frac{2}{5}$ ft. respectively, find the lengths of the sides, and the area of the triangle.

(29) The perimeter of an isosceles triangle, when the base is half the length of each of its sides, is 120 ft.; find the side of a square equal in area to the triangle.

(30*) A field is in the form of a right-angled triangle, the two sides containing the right angle being 100 and 200 yards; how many acres does it contain? And if the triangle be divided into two parts by a line drawn from the right angle perpendicular to the opposite side, what is the area of each part?

(31*) Find the area of an isosceles triangle whose base is 3 ft., and each of whose equal sides is 5 ft.

(32) The breadth of the gable-end of a house is 48 ft., and the distances of its eaves from the ridge are 35 ft. and 29 ft. respectively; find the area of the gable-top; and also the height of the roof.

(33) Find the cost of a triangular plot of land, whose sides measure, respectively, 87 yds., 100 yds., and 143 yds., 5*s.* 6*d.* per sq. yd.

(34) How many square feet of wood will be required for a triangular floor whose sides measure, respectively, 26 ft., 35 ft., and 51 ft.?

(35) The sides of a triangular field are 65 yds., 119 yds., and 138 yds. respectively; find the side of a square field containing the same amount of land.

(36) What is the area of a right-angled triangle (fig. 1), when the angle at C is 45°, and the base BC is 40 ft.?

(37) Find the area of a right-angled triangle (fig. 1), when the angle at C is 45°, and the hypothenuse is 98 ft.

(38) The angle at C in a right-angled triangle (fig. 1) is 30°, and the perpendicular height is 40 ft; find the area.

(39) The angle at C in a right-angled triangle (fig. 1) is 30°, and the hypothenuse is 60 ft.; find the area.

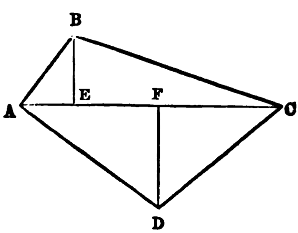
VI. THE TRAPEZIUM.

Definition.—The trapezium is a four-sided figure having none of its sides parallel to each other.

AC is the *diagonal*, and BE, FD the *perpendiculars* upon it.

RULE.—*To find the area, when the diagonal and the perpendiculars upon it are given.*

Multiply the sum of the perpendiculars by the diagonal, and divide the product by 2.



Note 1.—To find the sum of the perpendiculars, when the area and the diagonal are given.—Divide twice the area by the diagonal.

Note 2.—To find the diagonal, when the area and the sum of the perpendiculars are given.—Divide twice the area by the sum of the perpendiculars.

Note 3.—The trapezium ABCD is made up of the triangle ABC and triangle ADC: hence the area of the trapezium may be found by adding together the areas of these two triangles.

Note 4.—If the sides of a trapezium and its diagonal are given, then its area is the sum of the areas of the two triangles ABC and ADC; and these areas can be found by Rule 2, Prob. V.

Note 5.—If the sides of a trapezium and its diagonal, and also the distances AE and FC, at which the perpendiculars rise, are given, then BE and FD must first be found by the Rule 2 in Prob. I. Having found these perpendiculars, we shall then be able to apply the rule given above for finding the area of the trapezium.

Note 6.—If the trapezium is inscribed in a circle, that is, if its opposite angles are equal to two right angles, then its area will be found thus:—Add together the four sides, and take half their sum. From this half sum subtract each side separately. Multiply these four remainders together, and the square root of the product is the area of the trapezium.

Example 1.—Find the area of a field, in the shape of a trapezium, whose diagonal is 10 chains 20 links, and the perpendiculars let fall upon it from the opposite angles are 6 ch. 30 lks. and 4 ch. 10 lks.

Now, 10 ch. 20 lks. = 1020 lks.; 6 ch. 30 lks. = 630 lks.; 4 ch. 10 lks. = 410 lks.

Then the area $= \dfrac{\text{diagonal} \times \text{sum of perpendiculars}}{2}$

$$= \frac{1020 \times (630 + 410)}{2}$$

$$= \frac{1020 \times 1040}{2} = 530400 \text{ sq. lks.} = 5{\cdot}304 \text{ ac.}$$

$$= 5 \text{ ac. } 1 \text{ rood } 8\tfrac{16}{25} \text{ poles.}$$

Example 2.—If the area of a trapezium is 3250 sq. yds., and the diagonal is 100 yds., find the sum of the perpendiculars.

By Note 1, the sum of the perpendiculars $=\frac{2 \text{ area}}{\text{diagonal}}$

$$=\frac{2\times 3250}{100}=\frac{6500}{100}=65 \text{ yds.}$$

Formulæ:

I. $\text{Area}=\frac{\text{diagonal}\times\text{sum of perpendiculars}}{2}$

II. $\text{Diagonal}=\frac{2 \text{ area}}{\text{sum of perpendiculars}}$

III. $\text{Sum of perpendiculars}=\frac{2 \text{ area}}{\text{diagonal}}$

Examples.

Find the area of the trapezium, when the dimensions given are—

(1) The diagonal 48 ft., and its perpendiculars 24 ft. 6 in. and 15 ft. 6 in.

(2) The diagonal 34·5 ft., and its perpendiculars 19·75 ft. and 14·25 ft.

Find the area of a field, in the shape of a trapezium, when its dimensions are—

(3) The diagonal 400 yds., and its perpendiculars 120 yds. and 80 yds.

(4) The diagonal 862 lks., and its perpendiculars 380 lks. and 220 lks.

Find the diagonal of a trapezium, when its area and the sum of its perpendiculars are, respectively—

(5) Area 1134 sq. ft., and sum of the perpendiculars 42 ft.

(6) Area 5 ac. 3 roods 35 poles $3\frac{1}{4}$ sq. yds., and sum of the perpendiculars 233 yds.

(7) Area 6 ac. 1 rood 20 poles, and sum of the perpendiculars 12 ch. 50 lks.

(8) Find the cost of a plot of land, in the shape of a trapezium, whose diagonal is 108 ft., and the perpendiculars upon it 55 ft. 3 in. and 60 ft. 9 in. respectively, at 2*s*. 6*d*. per sq. yd.

(9) The area of a trapezium is 759½ sq. yds., its diagonal is 108 ft. 6 in., and one of its perpendiculars is 65 ft. 3 in.; find the other perpendicular.

(10) ABCD (see fig.) is a trapezium having AB=87 ft., BC = 119 ft., AD = 169 ft., CD = 41 ft., and the diagonal AC=200 ft.; find the area.

(11) Find the area of the four-sided figure ABCD (see fig.), whose dimensions are: AB = 145 ft., CD = 135 ft., AE = 87 ft., FC=81 ft., and the diagonal AC=368 ft.

(12) Find the area of a trapezium, inscribed in a circle, whose sides are, respectively, 48 ft., 52 ft., 56 ft., and 60 ft.

(13) Find the area of a trapezium whose opposite angles are together equal to two right angles, and whose sides are, respectively, 80 ft., 110 ft., 120 ft., and 150 ft.

(14*) The diagonal of a trapezium is 50·08 ft., and the perpendiculars upon it from the two opposite angles are 10·12 ft. and 8·4 ft.; find the area.

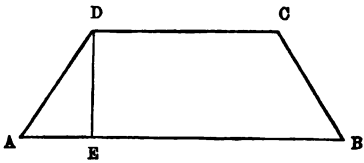
(15**) The length of the diagonal of a four-sided figure is 54 ft., and the lengths of the perpendiculars upon the diagonal from the opposite corners are 23 ft. 9 in. and 18 ft. 3 in.; how many square yards are there in the field?

(16**) ABCD is a quadrilateral field; AB = 48 chains, BC=20 chains, the diagonal AC=52 chains, and the perpendicular from D upon AC=30 chains. Find the area of the field.

(17**) One diagonal of a quadrilateral field is 10 chains 14 links, and the perpendiculars upon it from the angles are 6 chains 27 links, and 8 chains 6 links. How many acres does the field contain?

VII. THE TRAPEZOID.

Definition.—The trapezoid is a four-sided figure, having two opposite sides parallel.



AB is parallel to DC.

DE is the *perpendicular distance* between the parallel sides, or frequently called simply the *distance*.

RULE.—*To find the area of a trapezoid.*

Multiply the sum of the parallel sides by the perpendicular distance between them, and divide the product by 2.

Or, multiply half the sum of the parallel sides by the perpendicular distance between them.

Note 1.—To find the sum of the parallel sides, when the area and the perpendicular distance are given.—Divide double the area by the perpendicular distance.

Note 2.—To find the perpendicular distance, when the area and the sum of the parallel sides are given.—Divide double the area by the sum of the parallel sides.

Note 3.—Whenever we find the word *distance* mentioned in questions on the trapezoid, we must understand it to mean the *perpendicular distance* between the parallel sides.

Example 1.—Find the area of a field, in the shape of a trapezoid, whose parallel sides are 12 ch. 75 lks. and 8 ch. 25 lks., and the perpendicular distance between them is 6 ch. 20 lks.

Sum of the parallel sides=12 ch. 75 lks.+8 ch. 25 lks.
=21 ch.=2100 lks.

Then, area$=\frac{21 \text{ ch.} \times 6 \text{ ch. } 20 \text{ lks.}}{2}=\frac{2100 \times 620}{2}=651000$ sq. lks.=6 ac. 2 roods $1\frac{3}{5}$ poles.

Example 2.—The area of a field, in the shape of a trapezoid, whose parallel sides are 325 yds. and 215 yds., is 8 ac. 3 roods 28 poles 3 sq. yds.; find the perpendicular distance between them.

Now, the double of 8 ac. 3 roods 28 poles 3 sq. yds.= 17 ac. 3 roods 16 poles 6 sq. yds.=86400 sq. yds; and the sum of the parallel sides=325 yds.+215 yds.=540 yds.

Then, perpendicular distance$=\frac{86400}{540}=160$ yds.

$$\text{Formulæ}\begin{cases}\text{I. Area}=\dfrac{(\text{AB}+\text{CD})\times\text{DE}}{2}\\[2ex]\text{II. Sum of parallel sides}=\dfrac{2\text{ area}}{\text{DE}}\\[2ex]\text{III. Perpendicular distance}=\dfrac{2\text{ area}}{(\text{AB}+\text{CD})}\end{cases}$$

Examples.

Find the area of the trapezoid whose dimensions are, respectively—

(1) Parallel sides 50 ft. and 34 ft., and the perpendicular distance 25 ft.

(2) Parallel sides 24 ft. 9 in. and 15 ft. 3 in., and the perpendicular distance 12 ft. 6 in.

Find the area of a field, in the shape of a trapezoid, when the dimensions are—

(3) Parallel sides 256 yds. and 144 yds., and perpendicular distance 85 yds.

(4) Parallel sides 12 ch. 25 lks. and 7 ch. 75 lks., and ~endicular distance 10 ch. 40 lks.

Find the distance between the parallel sides in a trapezoid whose area and parallel sides are, respectively—

(5) Area 311 sq. yds. 1 sq. ft., and parallel sides 14 yds. 2 ft. and 12 yds.

(6) Area 3 ac. 0 rood 34 poles, and parallel sides 7 ch. 50 lks. and 5 ch.

(7) Find the rental of a field, in the shape of a trapezoid, when its parallel sides are 6 ch. 45 lks. and 3 ch. 55 lks., and the distance between them 4 ch. 20 lks., at £3 5*s.* per acre.

(8) The cost of a field, in the shape of a trapezoid, at £60 per acre, is £312 7*s.* 6*d.*; the parallel sides are 17 ch. and 12 ch. 75 lks. Find the distance between them.

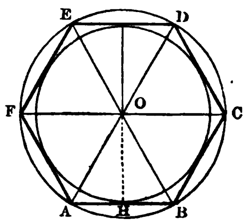
(9) The area of a field, in the shape of a trapezoid, is 2 ac. 2 roods 20 poles; the perpendicular distance between the parallel sides, of which one is 115 yds., is 70 yds. Find the other parallel side.

(10**) A field is bounded by four straight lines, of which two are parallel. If the sum of the parallel sides is 1235 lks., and the perpendicular distance between them is 240 lks., determine the area of the field.

(11) The rental of a field, which is in the shape of a trapezoid, is £9 8*s.* 6*d.*; the parallel sides are 200 yds. and 119 yds., and the perpendicular distance between them is 110 yds. Find the charge per acre.

VIII. THE REGULAR POLYGON.

Definitions.—A polygon is a figure contained by three or more sides; and it is also called regular when all its sides and angles are equal.



O is the centre of the polygon.

OH is the radius of the inscribed circle.

OA is the radius of the circumscribed circle.

The perimeter of a regular polygon is the length of each side multiplied by the number of sides.

If lines be drawn from centre O to the angles at A, B, C, &c., the figure will then be divided into equal triangles OAB, OBC, &c.

RULES.—(1) *To find the area of a regular polygon, when a side, and the perpendicular upon it from the centre, are given.*

Multiply the length of each side by the number of sides, and this product again by the perpendicular drawn from the centre to the middle of one of the sides; and half the product is the area of the polygon.

(2) *To find the area, when only the length of each side is given.*

Multiply the square of the side given by the number standing opposite the name of the polygon in the subjoined table; and the product is the area of the polygon.

TABLE OF MULTIPLIERS.

No. of Sides	Name	Area Multipliers	No. of Sides	Name	Area Multipliers
3	Trigon or Equi-triangle	·4330	8	Octagon	4·8284
4	Tetragon or square	1	9	Nonagon	6·1818
5	Pentagon	1·7205	10	Decagon	7·6942
6	Hexagon	2·5981	11	Undecagon	9·3656
7	Heptagon	3·6339	12	Dodecagon	11·1962

Note 1.—To find the length of each side, when the area and the perpendicular drawn from the centre to the middle of one of its sides are given.—Divide double the area by the perpendicular, and the quotient is the *sum* of all the sides. Again, divide the *sum* of all the sides by the *number* of sides in the polygon, and the quotient is the length of each side.

Note 2.—To find the length of each side when the area only is given.—Divide the area by the number standing opposite its name in the preceding table; then, the square root of this quotient is the length of the side.

Note 3.—In some cases it will be possible to find the area of a regular polygon when only one side is given, without using the table above and without recourse to Trigonometry. This can be effected in the case of a regular hexagon.

Since by Euclid (I. 15) all the angles at point O are equal to 360°,

therefore the angle AOB in the regular hexagon (see figure above) $= \frac{360°}{6} = 60°$.

And also since OA = OB,
therefore the angle OAB = angle OBA = 60°.

Hence, each of the three angles of the triangle OAB is

equal to 60°; and therefore the triangle OAB is equiangular, and consequently equilateral.

Therefore the area of a regular hexagon is $=6$ times the area of the equilateral triangle OAB, which can easily be found by Rule 2, Prob. V.

Note 4.—In the table which has been given, it will be seen that the multipliers for the area have been carried down to only four figures, but they will be found sufficiently accurate for all practical purposes.

Example 1.—Find the area of a regular hexagon, when each side measures 20 ft., and the perpendicular from the centre to the middle of one of the sides is 17·32 ft.

Now, the sum of the sides or perimeter of the figure $=6\times 20=120$ ft.

Then, area of hexagon $=\frac{1}{2}$ (perimeter $\times$ perpendicular)

$$=\tfrac{1}{2}\times 120\times 17{\cdot}32=1039{\cdot}2 \text{ sq. ft.}$$

Example 2.—Find the area of a regular octagon, each side of which is 30 ft.

Now, the multiplier standing opposite 'octagon' in the table is 4·8284.

Then, area of octagon $=$ square of the side multiplied by 4·8284

$$=30^2\times 4{\cdot}8284=900\times 4{\cdot}8284$$
$$=4345{\cdot}56 \text{ sq. ft.}$$

Formulæ
I. Area $=\dfrac{\text{perimeter}\times\text{perpendicular}}{2}$
II. Area $=\text{side}^2\times$ number standing opposite name of polygon.

EXAMPLES.

(1) Find the area of a trigon or equilateral triangle whose side is 2 ft. 6 in.

(2) Find the area of a regular hexagon, each side of which is 60 ft.

(3) Find the area of a regular heptagon, each side of which measures 30 ft.

(4) Each side of a regular octagon is 30 ft., and the perpendicular drawn from the centre to the middle of one of its sides is 36·213 ft.; find its area.

(5) The length of each side of a grass-plot, which is in the shape of a regular decagon, is 80 lks.; find the area.

(6) Each side of a regular heptagon measures 20 ft., and the perpendicular drawn from the centre to the middle of one of its sides is 20·764 ft.; find the area.

(7) Find the expense of walling-in a plot of land, in the shape of a regular hexagon, containing 1039·24 sq. yds., at 7*s.* 6*d.* per yd.

(8) An entrance hall, which measures 33 sq. yds. 4 sq. ft. 111·6 sq. in., is to be paved with mosaic-tiles, in the shape of a regular octagon, each side of which is 3 in.; how many will be required?

(9) The perimeter of a regular hexagon is 480 ft., and that of a regular octagon is the same; compare the areas of the two figures.

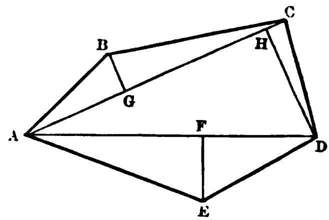
IX. THE IRREGULAR POLYGON.

Definition.—An irregular polygon is a figure contained by three or more sides, whose sides and angles are not equal.

RULE.—*To find the area of an irregular polygon.*

Divide the figure, in the most convenient manner, into triangles, trapeziums, &c. Then the area of the polygon is the sum of the areas of the different figures.

Example.—Find the area of the irregular polygon ABCDE (see fig.), when the following dimensions are given:—The diagonals AC and AD are 10 chains 30 links and 8 chains 15 links respectively, and the perpendiculars BG, HD, and EF are 4 chains 10 links, 6 chains 10 links, and 4 chains respectively.



Now, the figure ABCDE = trapezium ABCD + triangle ADE.

$$\text{Area of trapezium ABCD} = \frac{\text{AC} \times (\text{BG} + \text{HD})}{2}$$

$$= \frac{10 \text{ ch. } 30 \text{ lks.} \times 10 \text{ ch. } 20 \text{ lks.}}{2}$$

$$= 525300 \text{ sq. links};$$

$$\text{and area of triangle AED} = \frac{\text{AD} \times \text{EF}}{2} = \frac{8 \text{ ch. } 15 \text{ lks.} \times 4 \text{ ch.}}{2}$$

$$= 163000 \text{ sq. links.}$$

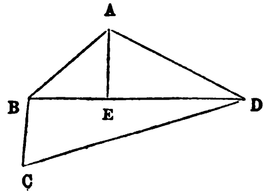
Then, area of polygon ABCDE

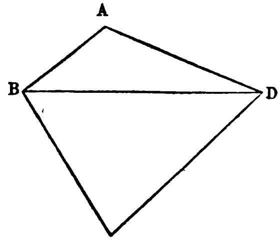
= 525300 sq. lks. + 163000 sq. lks. = 688300 sq. links.

= 6 ac. 3 roods 21 poles 8·47 sq. yds.

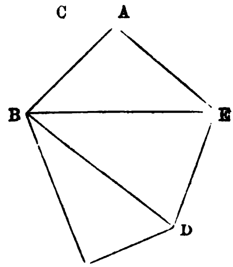
EXAMPLES.

(1) Find the area of the field ABCD, when the following measurements are given:—The diagonal BD is 10 chains 14 links; the perpendicular AE upon it 8 chains 5 links; and the side BC, which is perpendicular to BD, 6 chains 25 links.



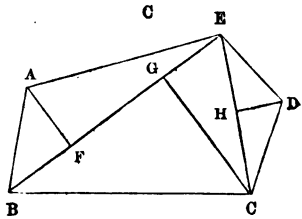


(2) In the field ABCD the following measurements are taken:—AB=39 yds., BC=143 yds., CD = 168 yds., AD=280 yds., and BD=305 yds. Find the area.



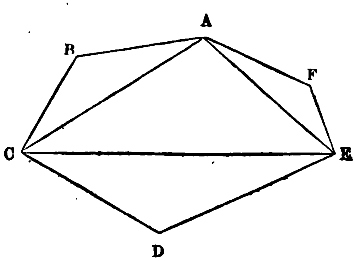
(3) Find the area of the field ABCDE, when the following measurements are given:—AB measures 61 yds., BC 140 yds., CD 23 yds., ED 91 yds., AE 69 yds.

Also the diagonal BD measures 159 yds., and the diagonal BE 100 yds.



(4) Find the area of the field ABCDE, when the following measurements are given:—The diagonal BE 108 yds., and the perpendiculars AF and CG upon it 49 yds. and 67 yds. respectively.

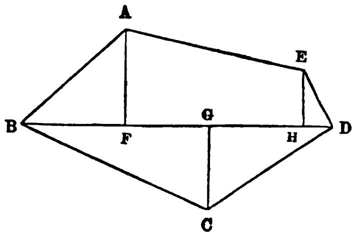
The diagonal EC measures 96 yds., and the perpendicular DH upon it 35 yds.



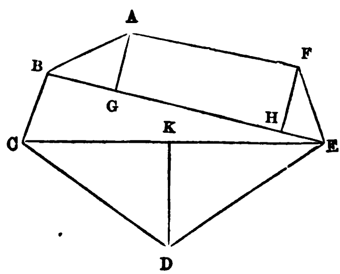
(5) Find the area of the field ABCDE, when the following dimensions are given:—AB = 85 yds., BC=76yds., CD=87yds., DE=100 yds., EF=17 yds., and AF=105 yds.

Also the diagonals are: AC=105 yds., CE=143 yds., and AE=116 yds.

(6) In taking measurements of the plot of land ABCDE, it is found that the diagonal BD measures 475 yds., and that the perpendicular AF, which rises at F, a distance of 175 yds. from B, measures 160 yds. Proceeding on to G, a distance of 320 yds. from B, it is found that the perpendicular CG is 160 yds., and then at H, a distance of 420 yds. from B, it is found that the perpendicular EH is 90 yds. Find the area.

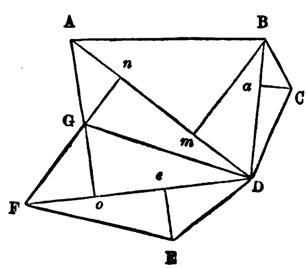


(7) Given the following measurements of the field ABCDEF; find the area. On the diagonal BE, which is 323 yds. long, the distance BG is 63 yds., and its perpendicular AG is 54 yds., and the distance BH is 283 yds., and the perpendicular FH is 60 yds.



The diagonal CE is 312 yds., and its perpendicular KD is 115 yds. The side BC is 125 yds.

(8) Find the area of the irregular field ABCDEFG, having the following dimensions:— AD=170 yds., nG=40 yds., mB=66 yds., BD=82 yds., aC=26 yds., DF=160 yds., eE=30 yds., and oG=56 yds.



X. OFFSETS.

Definitions.—The area of a long irregular figure can best be found by means of ordinates or offsets. Take any

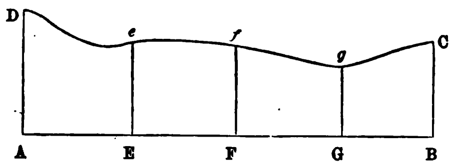


Fig. 1.

straight line AB, and at points A, E, F, G, &c, let AD, E*e*, F*f*, &c., be drawn perpendicular to AB. Then these *perpendiculars* AD, E*e*, F*f*, &c., are called *ordinates* or *offsets*.

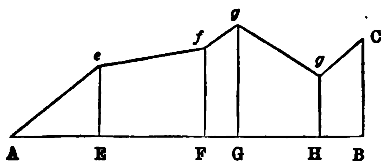


Fig. 2.

The offsets may be taken at *equal* distances AE, EF, FG, &c., as in fig. 1; and also at *unequal* distances, as in fig. 2, in which case the whole figure is divided into triangles and trapezoids.

RULES.—*To find the area of a long irregular figure.*

(1) *When the offsets are taken at equal distances* (fig. 1).—To half the sum of the first and last breadths, add all the intermediate breadths; divide the sum by the *number* of *equal* parts in the line AB (and not by the number of breadths), and the quotient is the *mean* breadth of the figure. Multiply the mean breadth by the length of the figure, and the product is the area, nearly.

(2) *When the offsets are taken at unequal distances* (fig. 2).—Find the areas of all the triangles and trapezoids into which the figure is divided; and their sum is the area, very nearly.

Note 1.—Another rule also may be given, which can be used in cases where great accuracy is not essential.—Add all the breadths together, and divide by the number of them for a mean breadth; multiply this mean breadth by the length of the figure, and the product is the area, nearly.

Note 2.—The rules in this Section will give the area a little *less* than it actually is, but sufficiently correct for all practical purposes; while the rule given in Note 1 gives the area *more* than it really is, and with less exactness than the other.

Note 3.—All the questions in this Section may be worked either by Rule 1 or Rule 2: the first rule, however, is generally employed when the ordinates are taken at equal distances; but when they are taken at unequal distances, it will be better to employ Rule 2.

Example 1.—The perpendicular breadths or offsets of any irregular figure (fig. 1) at five equidistant places A, E, F, &c., are 10 ft., 7 ft., 9 ft., 6 ft., and 8 ft., and its length is 30 ft.; find the area.

By Rule 1:

$$\begin{array}{rl}
 & 10 \text{ ft.} = \text{AD} \\
 & 8 \text{ ft.} = \text{BC} \\
2) & \overline{18} \\
 & \overline{9} \quad = \text{half the sum of extreme breadths} \\
 & \left.\begin{array}{r} 7 \\ 9 \\ 6 \end{array}\right\} = \text{intermediate breadths} \\
4) & \overline{31} \\
 & \overline{7\cdot75} = \text{mean breadth} \\
 & 30 \\
 & \overline{232\cdot5} \text{ sq. ft.} = \text{area, nearly.}
\end{array}$$

Example 2.—Find the area of an irregular plot of land (fig. 2) from the following offsets taken in yards:—At 12, 10; at 30, 16; at 40, 24; at 64, 16; at 84, 20.

Here the figure consists of one triangle and four trapezoids; hence the area of the figure is the sum of the areas of the triangle and trapezoids; and the question may be thus worked:—

$$\begin{array}{r} 12\times10=120 \\ 18\times(10+16)=468 \\ 10\times(16+24)=400 \\ 24\times(24+16)=960 \\ 20\times(16+20)=720 \\ \hline 2)\overline{2668} \\ \hline 1334 \end{array}$$
square yards—area.

EXAMPLES.

[Questions 1 and 2 are worked by Rule 1; all the rest by Rule 2.]

(1) Find the area of an irregular figure, when its length is 60 ft., and its breadths, taken at six equidistant places, are 24, 14, 16, 16, 18, and 22 ft.

(2) Find the area of an irregular plot of land, which is 40 yds. long, and its breadths, taken at six equidistant places, are 0, 4, 10, 16, 24, and 8 yds.

(3) Find the area of a plot of land from the following offsets, taken in yards:—At 12, 20; at 20, 18; at 50, 60; and at 80, 40.

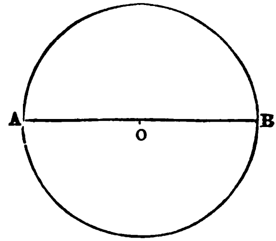
(4) In measuring a plot of land, the following are the offsets taken in yards:—At 0, 60; at 20, 38; at 35, 30; at 60, 16; and at 70, 0. Find the area.

(5) In measuring a plot of land, the following offsets are taken in links:—At 0, 120; at 120, 164; at 200, 106; at 260, 60; and at 300, 0. Find the area.

(6) Find the area of an irregular plot of land when its offsets in yards are: at 0, 0; at 30, 20; at 50, 34; at 100, 26; at 140, 40; and at 180, 20.

XI. THE CIRCLE.—THE CIRCUMFERENCE AND DIAMETER.

Definitions.—A circle is a plane figure bounded by a curved line called the *circumference* or *perimeter*, and is such, that all lines drawn from the centre to the circumference are equal.



AB is the *diameter* of the circle.

AO or BO is the *radius* of the circle.

RULES.—(1) *To find the circumference of a circle, when its diameter is given.*

Multiply the diameter by $\frac{22}{7}$; that is, multiply the diameter by 22, and divide the product by 7.

(2) *To find the diameter of a circle, when the circumference is given.*

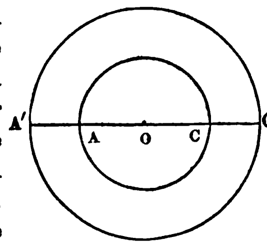
Divide the circumference by $\frac{22}{7}$; that is, multiply the circumference by 7, and divide the product by 22.

Note 1.—The above rules will give the answer with sufficient accuracy for all practical purposes. The length of the circumference, found by multiplying the diameter by $\frac{22}{7}$, will not be wrong to the one-hundredth part of the length of the radius. For instance, if the radius of a circle is 100 ft., then its circumference, obtained by multiplying the diameter by $\frac{22}{7}$, will not be 1 ft. wrong.

Note 2.—If still greater accuracy is desirable, then, to find the circumference, multiply the diameter by 3·1416 or by $\frac{355}{113}$; and to find the diameter, divide the circumference by 3·1416 or by $\frac{355}{113}$.

Note 3.—The *circumferences* of circles have the same ratio to each other as their *radii* or *diameters*. Thus, if the *diameter* of a circle is double that of another, then its *circumference* is also double that of the other; and if its *diameter* is three times as long, then its *circumference* is three times that of the other; and so on.

Note 4.—Concentric circles are those which are drawn from a common centre. And in questions which involve the finding of the space A A′ between two concentric circles, when their circumferences are given, we must find the diameters A C and A′ C′ of the two circles by Rule 2. Then, half the difference of these two diameters is the width A A′.



Example 1.—Find the circumference of a circle whose diameter is 11·27 ft.

The circumference $=$ diameter $\times \frac{22}{7} = 11{\cdot}27 \times \frac{22}{7} = \frac{247{\cdot}94}{7}$ $= 35{\cdot}42$ ft.

Example 2.—The circumference of a circle is 37 ft. 5 in.; find its diameter.

Now, 37 ft. 5 in. $=449$ in.

Then the diameter $= 449 \div \frac{22}{7} = 449 \times \frac{7}{22} = \frac{3143}{22} = 142\frac{19}{22}$ in. $=11$ ft. $10\frac{19}{22}$ in.

Example 3.—The fencing of a circular plot of land, at 4*s*. 2*d*. per yd., costs £59 11*s*. 8*d*.; find its diameter.

First, £59 11*s*. 8*d*. $\div$ 4*s*. 2*d*. gives 286 yards of fencing, which is also the circumference of the circle.

Then, the diameter $=$ circumference $\div \frac{22}{7} = 286 \div \frac{22}{7} = 286 \times \frac{7}{22} = 91$ yds.

Formulæ $\begin{cases} \text{I. Circumference} = \text{diameter} \times \frac{22}{7}. \\ \text{II. Diameter} = \text{circumference} \times \frac{7}{22}. \end{cases}$

EXAMPLES.

Find the circumference of a circle whose diameter is—

(1) Diameter 7 in. (2) Diameter 7 ft. 7 in.
(3) Diameter 81 ft. 8 in. (4) Diameter 51 yds. 0 ft. 5 in.
(5) Diameter 1 fur. 16 poles. (6) Diameter 9 ch. 24 lks.
(7) Diameter 35·49 ft. (8) Diameter 119·07 ft.

Find the diameter of a circle whose circumference is—

(9) Circumference 12 ft. 10 in.
(10) Circumference 36 ft. 8 in.
(11) Circumference 61 yds. 0 ft. 4 in.
(12) Circumference 6 fur. 24 poles.
(13) Circumference 3 ch. 30 lks.
(14) Circumference 38·5 ft.
(15) Circumference 314·16 ft.

(16) The diameter of a carriage wheel is $3\frac{1}{2}$ ft.; how many revolutions does it make in traversing one-fourth of a mile?

(17) If a carriage wheel makes 220 revolutions in traversing half a mile, find its diameter.

(18) There are two concentric circles: the circumference of the inner circle is 16 ft. 6 in., and of the outer one is 18 ft. 4 in. Find the width of the ring.

(19) There are two concentric circles: the circumference of the outer circle is 440 ft., and of the inner one is 330 ft. Find the width of the ring.

(20) The walling-in of a circular plot of land, at 12*s*. 6*d*. per yd., costs £123 15*s*.; find its diameter.

(21) In raising water from the bottom of a well by means of a wheel, it is found that the wheel, whose diameter is 2 ft. 4 in., makes 30 revolutions in raising the bucket; find the depth of the well.

(22**) If the diameter of a well is 3 ft. 9 in., what is its circumference?

(23**) Find the radius of a circle whose perimeter is 100 chains.

(24) The minute-hand of a clock is 5 in. long; find the length of the circle that its point will describe in an hour's time.

XII. THE AREA OF A CIRCLE.

Definitions.—See Problem XI.

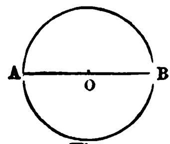


Fig. 1.

Rules.—(1) *To find the area of a circle, when its radius or diameter is given.*

Multiply the square of the radius by $\frac{22}{7}$; or, multiply the square of the diameter by $\frac{11}{14}$.

(2) *To find the area of a circle, when its circumference is given.*

Multiply the square of the circumference by $\frac{7}{88}$.

Note 1.—To find the diameter or radius of a circle, when its area is given.—Divide the area by $\frac{11}{14}$, and the square root of the quotient is the diameter; or, divide the area by $\frac{22}{7}$, and the square root of the quotient is the radius.

Note 2.—To find the circumference of a circle, when its area is given.—Divide the area by $\frac{7}{88}$, and the square root of the quotient is the circumference.

Note 3.—The above rules give the area of a circle with sufficient accuracy for all practical purposes; but whenever greater accuracy is desirable, then the following rules may be used for finding the area of a circle:—Multiply the square of the diameter by ·7854, and the product is the area; or

multiply the square of the circumference by ·07958, and the product is the area.

Note 4.—The area of a circle is equal to that of a triangle whose base is equal to the circumference, and whose perpendicular is equal to the radius of the circle.

Note 5.—Of all plane figures, the circle is that which contains the greatest area within the *same* perimeter; that is, if we have a square, a rectangle, a circle, or any other plane figure, all of which have the *same* perimeter, the circle will contain the greatest area.

Note 6.—The diameter of a circle inscribed in a square, that is, of a circle which touches each side of a square ABCD, is EF.

And EF=AB=CD.

Hence, the diameter of an inscribed circle is equal to a side of the square.

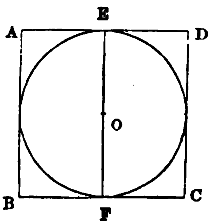


Fig. 2.

Note 7.—To find the side AB of a square inscribed in a circle ABCD.—Since AOB is a right-angled triangle, having two equal sides AO, OB (each of which is half the diameter of the circle), we shall be able to obtain AB, the hypothenuse, by Rule 1, Prob. I.

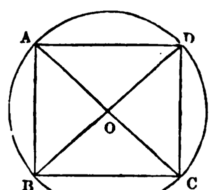


Fig. 3.

Or thus: the area of the square$=\frac{AC \times BD}{2}$ (see Note 4, Prob. IV.)$=\frac{\text{diameter}^2}{2}$.

Having thus found the area of the inscribed square, its *side* will be found by taking the square root of the area.

Note 8.—To find the side of a square that is equal in area to a given circle.—In this case, we must first find the area of the given circle; and the square root of this area is side of the required square.

Note 9.—To find the area between two concentric circles ABC and DEF.—The area between the two circles is the difference between the area of the larger circle ABC and of the smaller circle DEF; and these areas can easily be found by the rules given in this chapter.

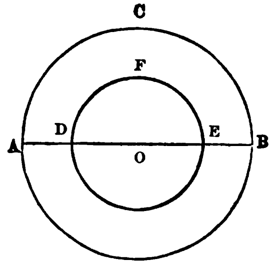


Fig. 4.

Or, the area may be found by either of the following rules:—

(*a*) Multiply the sum of the radii of the two circles by their difference, and the product by $\frac{22}{7}$. (*b*) Multiply the sum of the outer and inner diameters by their difference, and the product by $\frac{11}{14}$.

Note 10.—The *areas* of circles are proportional to the *squares* of their *radii*, or to the *squares* of their *diameters*.

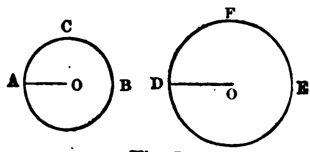


Fig. 5.

Thus,

Area of smaller circle : larger circle $:: \frac{22}{7} \times OA^2 : \frac{22}{7} \times OD^2$
$:: OA^2 : OD^2$.

In the same manner, it may be shown that their areas are proportional to their *diameters*.

(*a*) For instance, if the radius (AO) of the smaller circle is 10 ft., and it is required to find the radius (OD) of the larger circle, whose area is twice that of the smaller circle, we shall have the following proportion:—

$OD^2 : OA^2 ::$ area of larger circle : area of smaller circle.

Substituting the known terms, we have

$$OD^2 : 10^2 :: 2 : 1;$$

therefore $OD^2 = 2 \times 10^2$.

And also the square roots of these quantities are equal; therefore $OD = 10\sqrt{2} = 10 \times 1{\cdot}414 = 14{\cdot}14 +$ ft.

Hence, if the area of the required circle is double that of the given circle, then its radius is found by multiplying the radius of the given circle by $\sqrt{2}$.

If the area is to be three times as much, then its radius is equal to the radius of the given circle multiplied by $\sqrt{3}$; and so on.

But if the area of the required circle is only *half* that of the given circle, then its radius is found by *dividing* the given radius by $\sqrt{2}$; and if the area is *one-third*, then its radius is found by *dividing* the given radius by $\sqrt{3}$; and so on.

[N.B. $\sqrt{2}=1{\cdot}4142$; $\sqrt{3}=1{\cdot}732$; $\sqrt{4}=2$, &c.]

(*b*) Also the method adopted in Note 5, *b*. Prob. II., may be used in the case of a circle, as well as in that of a square.

Note 11.—To find the radii (OG and OD) of two concentric circles which divide the circle ABC into three equal parts.

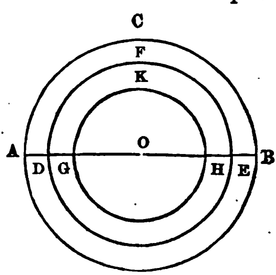


Fig. 6.

(*a*) Now, the area of the circle GHK$=\frac{1}{3}$ area of circle ABC.

Therefore the radius OG (Note 10)$=\text{OA}\times\sqrt{\frac{1}{3}}=\dfrac{\text{OA}}{\sqrt{3}}$.

That is, the radius OG=radius of the whole circle divided by the square root of 3.

Also, the area of the circle DEF $=\frac{2}{3}$ area of circle ABC.

Therefore the radius OD (Note 10)$=\text{OA}\times\sqrt{\frac{2}{3}}$.

That is, the radius OD=radius of the whole circle multiplied by $\sqrt{\frac{2}{3}}$.

(*b*) But a question like the preceding may be worked in the following manner:—

First, find the area of the circle ABC, whose radius is

given; and $\frac{1}{3}$ of this area is the area of the circle GHK. Then the radius of this circle GHK may be found by Note 1.

Again, the area of the circle DEF is $\frac{2}{3}$ of the area of the circle ABC. And having thus got the area of the circle DEF, its radius can be found by Note 1.

We shall thus have found the two radii OG and OD.

Example 1.—Find the area of a circle whose diameter is 11 ft.

Area of circle $=$ diameter$^2 \times \frac{11}{14} = 11^2 \times \frac{11}{14} = 121 \times \frac{11}{14} = \frac{1331}{14} = 95\frac{1}{14}$ sq. ft. $=$ 95 sq. ft. $10\frac{2}{7}$ sq. in.

Example 2.—The area of a circle is 52 sq. ft. 58 sq. in.; find its diameter.

Now, 52 sq. ft. 58 sq. in. $=$ 7546 sq. in.

Then, diameter $= \sqrt{7546 \div \frac{11}{14}} = \sqrt{7546 \times \frac{14}{11}} = \sqrt{9604} =$ 98 in. $=$ 8 ft. 2 in.

Example 3.—A gravel walk, which is 1 yd. wide, runs round the outside of a circular plot of land, whose diameter is 60 ft.; find the area of the walk.

The diameter DE (fig. 4) $=$ 60 ft.

And diameter AB $=$ DE $+$ AD $+$ EB $=$ DE $+ 2$ width of walk $=$ 60 ft. $+ 2$ yds. $=$ 66 ft.

Then (Note 9, *a*), sum of the radii of two circles $= 33 + 30 = 63$; and difference of two radii $= 33 - 30 = 3$.

Therefore area of walk $= 63 \times 3 \times \frac{22}{7} = 594$ sq. ft.

Example 4.—The diameter of a circle is 30 in.; find the diameter of *another* circle whose area is double that of the given circle.

Since the area of the required circle is double that of the given circle,

therefore (by Note 10, *a*) the diameter of the *required* circle $=$ diameter of *given* circle $\times \sqrt{2}$

$$= 30 \times \sqrt{2} = 30 \times 1{\cdot}414 +$$
$$= 42{\cdot}42 + \text{in.}$$

Or, by Note 10, *b*, the area of given circle$=30^2\times\frac{11}{14}=\frac{4950}{7}$ sq. in.$=707\frac{1}{7}$ sq. in.

Therefore area of required circle=area of given circle $\times 2=707\frac{1}{7}\times 2=1414\frac{2}{7}$ sq. in.

And its diameter$=\sqrt{1414\frac{2}{7}\times\frac{14}{11}}=\sqrt{1800}=42{\cdot}42+$in.

$$\text{Formulæ}\begin{cases}\text{I. Area}=\text{radius}^2\times\frac{22}{7}=\text{diameter}^2\times\frac{11}{14}\\ \qquad\qquad=\text{circumference}^2\times\frac{7}{88}.\\ \text{II. Diameter}=\sqrt{\text{area}\times\frac{14}{11}}.\\ \text{III. Radius}=\sqrt{\text{area}\times\frac{7}{22}}.\\ \text{IV. Circumference}=\sqrt{\text{area}\times\frac{88}{7}}.\end{cases}$$

EXAMPLES.

Find the area of the circle whose diameter is—

(1) Diameter 1 ft. 2 in. (2) Diameter 15 ft. 2 in.
(3) Diameter 22 ft. 2 in. (4) Diameter 4 yds. 2 ft.
(5) Diameter 15·4 yds. (6) Diameter 18 yds. 2 ft.
(7) Diameter 3 ch. 22 lks. (8) Diameter 9·8 ft.

Find the area of the circle whose circumference is—

(9) Circumference 7 ft. 4 in. (10) Circumference 44 ft.
(11) Circumference 14 ft. 8 in.
(12) Circumference 264 yds.
(13) Circumference 2 miles.

Find the diameter of the circle whose area is—

(14) Area 616 sq. in. (15) Area 1095 sq. yds. 1 sq. ft.
(16) Area 2 roods 1 pole $13\frac{3}{4}$ sq. yds.
(17) Area 1 ac. 2 roods 9 poles $13\frac{3}{4}$ sq. yds.
(18) Area 6 acres.

Find the circumference of the circle whose area is—

(19) Area 38 sq. ft. 72 sq. in.
(20) Area 2 ac. 2 roods 12 poles 11 yds.
(21) Area 1 ac. 4104 sq. lks.
(22) Area 2 acres 176 sq. yds.

(23) Find the area of a semicircle, when the radius of the circle is 14 ft.

(24) Find the number of square yards in a quadrant, when the radius of the circle, of which the quadrant is part, is 7 ft.

(25) The diameter of a circle is 56 ft.; find the length of the side of a square equal in area to the circle.

(26) The area of a square is 196 sq. ft.; a circle is inscribed in the square touching each side of it. Find the area of the circle.

(27) Find the cost of enclosing a circular garden, containing 2 roods 1 pole $13\frac{3}{4}$ sq. yds., with a wall, at 15*s.* 6*d.* per yd.

(28) A circular fish-pond, which covers an area of 3 ac. 0 rood 29 poles $2\frac{1}{4}$ sq. yds., is surrounded by a walk 3 yds. wide; find the expense of gravelling the walk at 4*d.* per sq. yd.

(29) All round a circular plot of land, which contains 2 ac. 2 roods 12 poles 11 sq. yds., is to be constructed a walk of uniform width; what must be its width that it may cover exactly $\frac{1}{4}$ of an acre?

(30) The diameter of a circular grass-plot is 28 ft.; find the diameter of *another* circular grass-plot which is double the size.

(31) A square piece of wood is 5 ft. 10 in. long; out of it is cut the greatest possible circle. Find how many square inches of wood are cut away.

(32) Out of a circular piece of wood, whose diameter is 3 ft. 4 in., is cut the largest possible square; find the length of its side.

(33) The largest possible circle, whose area is 17 sq. ft. 16 sq. in., is cut out of a square piece of wood; find how much larger the square is than the circle.

(34) The side of a square is 11 ft.; find the diameter of a circle equal in area.

(35) The diameter of a circle is 7 ft.; find the side of a square that has the same area as the circle.

(36) Find the area between two concentric circles, when the circumference of the outer circle is 440 ft. and of the inner one 352 ft.

(37) Find the expense of gravelling a walk, 2 yds. wide, running round the outside of a circular shrubbery, whose diameter is 70 yds., at 10*d.* per sq. yd.

(38) Find the area of a circular ring, 4 ft. wide, when the diameter of the outer circle is 64 ft.

(39) In a square room, whose side measures 17 ft. 6 in., is to be dug a circular bath, whose circumference touches the walls of the room. Find the area of the bath, and also the space left in the corners.

(40) The perimeter of a circle, of a square, and of an equilateral triangle is 132 ft.; compare the areas of these three figures.

(41*) A circular grass-plot, whose diameter is 40 yds., contains a gravel walk, 1 yd. wide, running round it 1 yd. from the edge. What will it cost to turf the gravel walk, at 4*d.* per sq. yd?

(42**) What is the expense of paving a circular court 30 ft. in diameter, at 2*s.* 3*d.* per sq. ft., leaving in the centre an hexagonal space, each side of which measures 2 ft.?

(43**) A room, 25 ft. 3 in. long and 14 ft. 6 in. wide, has a semicircular bow 21 ft. in diameter thrown out at one side; find the area of the whole room.

(44**) A circle 18 ft. in diameter is divided into three parts by two concentric circles; find the lengths of their radii, so that the inner circle and two rings may all be equal to one another.

(45*) A square courtyard has a circular basin in the middle of it, which is 13 ft. in diameter, a side of the court being 36 ft.; find what it will cost to gravel it, at ½*d.* a square foot—a flower-bed, 4 ft. wide, being left round three sides.

(46) A circular field, containing 10 ac. 1 rood 9 poles 13¾ sq. yds., is to be enclosed with a fence all round, which 1*s.* 9*d.* per yd.; find the expense.

(47) A gardener has a circular garden plot, containing 3 roods 7 poles $8\frac{1}{4}$ sq. yds., which he desires to plant with three different kinds of trees, and all occupying the same extent of ground; one kind being planted so as to form a circle in the middle; round this a belt of a different kind; and outside of this another belt containing a third kind. What is the diameter of the middle part, and what is the width of the two belts?

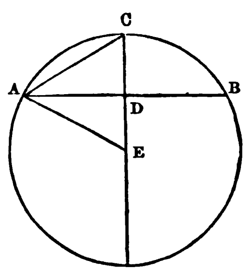
(48) A circular-fish pond, whose diameter is 98 yds., is to be planted all round it to an uniform width, so that the plantation may contain the same area as the fish-pond; find the width of the plantation.

(49*) A gravel walk, 4 ft. wide, is made round a circular court 90 yds. in diameter; the making of the path costs 8*d.* per sq. yd., and a border along its inner edge costs 3*d.* a yard. Find the total cost.

(50**) Calculate the expense of making a moat round a circular island, at 2*s.* 6*d.* per square yd.; the diameter of the island being 525 ft., and the breadth of the moat being 21 ft. 6 in.

XIII. THE CHORDS OF A CIRCLE.

Definition.—Any part of the circumference of a circle, as ACB, is called the *arc* ACB.



AB is the *chord* of the arc ACB.

AD or BD is *half the chord* of the arc ACB.

AC is the *chord of half the arc* ACB.

CD is the *height* or *versed sine* of the arc ACB.

CF is the diameter of the circle, and E the centre of the circle.

N.B.—The following rules are derived from these tw

important formulæ, which, if thoroughly impressed upon the mind, will enable the student to work the questions in this chapter without referring to the rules given below:—

$$AD^2 = FD \times CD;$$
$$AC^2 = CD \times CF.$$

RULES.—(1) *To find* (CF) *the diameter of a circle, when* (AB) *the chord of the arc, and* (CD) *the height of the arc are given.*

Divide the square of (AD) half the chord by (CD) the height of the arc; and the quotient will be (DF) the part of the diameter wanting; to which add (CD) the height of the arc, and the sum will be the diameter (CF).

(2) *To find* (CF) *the diameter of a circle, when* (AC) *the chord of half the arc and* (CD) *the height are given.*

Divide the square of (AC) the chord of half the arc by (CD) the height of the arc, and the quotient is (CF) the diameter of the circle.

(3) *To find* (AB) *the chord, when* (CF) *the diameter and* (CD) *the height of the arc are given.*

Multiply together the two parts (CD and DF) into which the diameter (CF) is cut by the chord (AB); and the square root of this product gives (AD or BD) half the chord of the arc. The double of this will be (AB) the chord of the arc.

(4) *To find* (AC) *the chord of half the arc, when* (CD) *the height of the arc and* (CF) *the diameter are given.*

Multiply together (CF) the diameter and (CD) the height of the arc; and the square root of the product is (AC) the chord of half the arc.

(5) *To find* (CD) *the height of the arc, when* (AC) *the chord of half the arc, and* (CF) *the diameter, are given.*

Divide the square of (AC) the chord of half the arc by (CF) the diameter of the circle; and the quotient is (CD) the height or versed sine of the arc.

Note 1.—The formula $AD^2 = DF \times CD$ can be thus proved: In the figure given above, because ADE is a right-angled triangle, we have (see Rule 2, Prob. I):

$AD^2 = AE^2 - DE^2 = EF^2 - DE^2$ (since $EF = AE$, being radii of the circle)

$= (EF + DE)(EF - DE) = FD(EC - DE)$, since $EF = EC$; therefore $AD^2 = FD \times CD$.

Note 2.—The formula $AC^2 = CD \times CF$ can be proved in the following manner:—

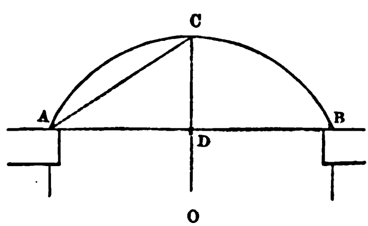
Since ADC (see figure given) is a right-angled triangle, we have

$$AC^2 = AD^2 + CD^2.$$

But in Note 1, it has been proved that $AD^2 = FD \times CD$. Substituting this, we have

$$AC^2 = (FD \times CD) + CD^2 = CD(FD + CD) = CD \times CF.$$

Note 3.—The arch of a bridge is frequently in the shape of the arc of a circle, in which case



CD is the rise, or versed sine, or height above the piers.

AB is the span or chord of the arch.

C, the middle point of the arch, is the crown.

AC is the distance of the spring of the arch to the crown C, or the chord of half the arch.

OC is the radius with which the arch was drawn.

Example 1.—The chord of an arc is 30 inches, and its height is 10 inches; find the diameter of the circle.

Now, by Rule 1, (FD) part of the diameter wanting

$=\text{chord}^2 \div \text{height}.$

$=30^2 \div 10 = 900 \div 10 = 90$ inches.

Then diameter $= (\text{FD}+\text{CD}) = 90+10 = 100$ in. $=$ 8 ft. 4 in.

Example 2.—The chord of half an arc is 12 ft. 8 in., and the height of the chord is 1 ft. 7 in.; find the diameter of the circle.

Now, 12 ft. 8 in. $=152$ in.; and 1 ft. 7 in. $=19$ in.

Then, by Rule 2, diameter $=$ square of chord of half the arc divided by height

$$=152^2 \div 19 = 23104 \div 19 = 1216 \text{ in.} = 101 \text{ ft. } 4 \text{ in.}$$

Example 3.—The height of an arc is 16 ft., and the diameter of the circle is 65 ft.; find the chord of the arc.

The two sections (CD, DF) into which the diameter is divided are 16 and $(65-16)$, or 16 and 49 ft.

Then by Rule 3, AD, half the chord of the arc

$$=\sqrt{\text{CD} \times \text{DF}} = \sqrt{16 \times 49} = \sqrt{784} = 28 \text{ ft.}$$

Therefore AB, the chord of the arc, $=2 \times 28 = 56$ ft.

Example 4.—The height of an arc is 36 ft., and the diameter of the circle is 196 ft.; find the chord of half the arc.

By Rule 4: AC, the chord of half the arc, $=$ square root of the product of the diameter and height.

$$=\sqrt{196 \times 36} = \sqrt{7056} = 84 \text{ ft.}$$

Example 5.—The chord of half an arc is 36 ft., and the diameter of the circle is 108 ft.; find the height of the arc.

By Rule 5: The height of the arc $=$ chord of half the arc$^2 \div$ diameter

$$=36^2 \div 108 = 1296 \div 108 = 12 \text{ ft.}$$

Formulæ:

Principal: (I.) $AD^2 = DF \times CD$.

(II.) $AC^2 = CD \times CF$.

Derived from above (III.) $CF = \frac{AD^2}{CD} + CD$.

(IV.) $CF = \frac{AC^2}{CD}$. (V.) $CD = \frac{AC^2}{CF}$.

(VI.) $AC = \sqrt{CD \times CF}$.

(VII.) $AB = 2AD = 2\sqrt{CD \times DF}$.

Examples.

(1) The chord of half an arc is 25 ft., and the height of the arc is 15 ft.; find the diameter of the circle.

(2) The chord of half an arc is 17 ft., and the height of the arc is 7 ft.; find the diameter of the circle.

(3) The chord of half an arc is 17 ft. 9 in., and the height of the arc is 6 ft. 3 in.; find the diameter of the circle.

(4) The chord of half an arc is 25·5 ft., and the versed sine of the arc is 5·1 ft.; find the diameter of the circle.

(5) The chord of half an arc is 15 ft., and the diameter of the circle is 40 ft.; find the height of the arc.

(6) The chord of half an arc is 12 ft., and the diameter of the circle is 36 ft.; find the height of the arc.

(7) The chord of half an arc is 28 ft., and the diameter of the circle is 56 ft.; find the height of the arc.

(8) The chord of half an arc is 60 ft., and the diameter of the circle is 125 ft.; find the height of the arc.

(9) The height of an arc is 9 in., and the diameter of the circle is 25 in.; find the chord of half the arc.

(10) The height of an arc is 2 ft. 1 in., and the diameter of the circle is 14 ft. 1 in.; find the chord of half the arc.

(11) The height of an arc is 4·9 ft., and the diameter of the circle is 12·1 ft.; find the chord of half the arc.

(12) The chord of half an arc is 17·5 ft., and the diameter of the circle is 175 ft.; find the height of the arc.

(13) The chord of an arc is 24 in., and the height of the arc is 9 in.; find the diameter of the circle.

(14) The chord of an arc is 4 ft. 4 in., and the height of the arc is 1 ft. 1 in.; find the diameter of the circle.

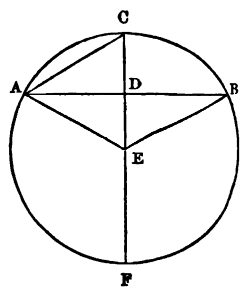
(15) The span (chord) of a bridge, the form of which is an arc of a circle, is 78 ft., and its height above the stone piers is 12 ft.; find with what radius it was described.

(16**) The span of a bridge, the form of which is an arc of a circle, being 96 ft., and its height being 12 ft., with what radius was it described?

(17) The height of the arch of a bridge, the form of which is an arc of a circle, is 24 ft., and the radius with which it is described is 312 ft.; find the span of the arch.

XIV. THE ARC OF A CIRCLE.

Definitions.—An arc of a circle is any part of the circumference.



Thus ACB is an *arc* of a circle less than a semicircle.

And AFB is an *arc* greater than a semicircle.

The angle AEB is the *angle* subtended at the centre by the arc ACB.

The circumference of any circle contains 360°.

The arc ACB bears the same ratio to the circumference of the circle that the angle AEB does to 360°.

RULES.—(1) *To find the length of the arc of a circle, when the number of degrees in the angle subtended by the arc at the centre, and the radius of the circle, are given.*

The radius of the circle being given, find its circumference by Rule 1, Prob. XI.

Then we have the following proportion for finding the length of the arc :—

360° : number of degrees in angle at centre :: circumference : required length of arc.

(2) *To find the number of degrees in the angle subtended by the arc at the centre of a circle, when the radius of the circle and the length of the arc are given.*

The radius of circle being given, find its circumference by Rule 1, Prob. XI.

Then we have the following proportion for finding the number of degrees in the angle at the centre:—

Circumference of circle : length of arc :: 360° : number of degrees subtended by arc.

(3) *To find the length of an arc of a circle less than a semicircle, when the chord of the arc and the diameter of the circle are given.*

From 8 times the chord of half the arc subtract the chord of the whole arc; divide the remainder by 3, and the quotient is the length of the arc, nearly.

Note 1.—The answers obtained by Rule 3 will not be quite correct, but they will be *sufficiently* correct in all cases when the arc is *less* than a semicircle.

Note 2.—When it is required to find the length of an arc AFB, *greater* than a semicircle, it will be better to find

first, the length of the remaining arc ACB by the preceding rules.

Then the length of the arc AFB = circumference of circle − arc ACB.

Note 3.—If it is desirable to find the length of the arc with still greater accuracy than is possible by Rule 3, we may proceed thus:—

If the diameter is not given, find it by the rules given in the last chapter. Then subtract $\frac{41}{60}$ of the height of the arc from the diameter; divide $\frac{2}{3}$ of the height by the remainder; and to the quotient add 1. Then multiply this sum by the chord of the arc, and the product is the length of the arc, very nearly.

Example 1.—The circumference of a circle is 50 inches, and the angle subtended at the centre by the arc is 30°; find the length of the arc.

By Rule 1, we have:

$360° : 30° :: 50$ in. : required length of the arc;

and from this proportion we get $4\frac{1}{6}$ in.—length of the arc.

Example 2.—The radius of a circle is 28 inches, and the angle subtended at the centre by the arc is 45°; find the length of the arc.

First, the circumference of the circle, by Rule 1, Prob. XI., is $= 56 \times \frac{22}{7} = 176$ in.

Then we have, by Rule 1:

$360° : 45° :: 176$ inches : required length of the arc.

From this proportion we get 22 in.—length of the arc.

Example 3.—The arc of a circle is 10 in.; the radius of the circle is 12 in. Find the angle subtended at the centre by the arc.

The circumference of the circle $= \frac{24}{1} \times \frac{22}{7} = 75\frac{3}{7}$ in.

Then by Rule 2, we have:

$75\frac{3}{7}$ inches : 10 inches :: 360° : required angle subtended by the arc.

From this proportion we obtain $47\frac{8}{11}$°, the angle subtended at the centre.

Example 4.—The chord of an arc less than a semicircle is 20 ft., and the chord of half its arc is 10·198 ft.; find the length of the arc.

By Rule 3: $10{\cdot}198 \times 8 = 81{\cdot}584$

Subtract 20

$3)\overline{61{\cdot}584}$

$\overline{20{\cdot}528}$ ft., length of arc, nearly.

Example 5.—Find the length of the arc of a circle less than a semicircle; the chord of the arc is 20 ft., and the diameter of the circle is 29 ft.

In the right-angled triangle ADE (see the figure) we have

$$DE = \sqrt{AE^2 - AD^2} = \sqrt{14{\cdot}5^2 - 10^2} = 10{\cdot}5 \text{ ft.};$$

and $CD = CE - DE = 14{\cdot}5 - 10{\cdot}5 = 4$ ft.

Again, in the right-angled triangle ADC, we have

AC, chord of half the arc, $= \sqrt{AD^2 + DC^2} = \sqrt{10^2 + 4^2} = \sqrt{116} = 10{\cdot}77$ ft.

Then, by Rule 3, we have:

$10{\cdot}77 \times 8 = 86{\cdot}16$

Subtract 20

$3)\overline{66{\cdot}16}$

$\overline{22{\cdot}05}$, length of the arc, nearly.

Formulæ:

I. 360° : number of degrees in angle at centre :: circumference of circle : required length of arc.

II. Circumference of circle : length of the arc :: 360° : number of degrees in angle at centre.

III. $\text{Arc} = \dfrac{8\,AC - AB}{3}$.

Examples.

(1) The radius of a circle is 10·5 in.; the angle subtended by the arc at the centre is 60°; find the length of the arc.

(2) The diameter of a circle is 35 in.; the angle subtended by an arc at the centre is 36°; find the length of the arc.

(3) The arc of a circle is 6 ft. 5 in.; the radius of the circle is 8 ft. 2 in.; find the angle subtended at the centre by the arc.

(4) The radius of a circle is 7 in.; the length of an arc is the same; find the angle at the centre subtended by the arc.

(5) The radius of a circle is 5 ft. 3 in.; find the whole perimeter of a sector, the angle of which is 45°.

(6) The arc of a circle is 5 ft. 6 in.; the angle subtended by the arc at the centre is 72°; find the radius of the circle.

(7) The arc of a circle is 26·4 ft.; the angle subtended by the arc at the centre is 36°; find the radius of the circle, of which the arc is a part.

(8) The radius of a circle is 84 ft.; the angle subtended at the centre by an arc is 11° 15′; find the length of the arc.

(9) The chord of an arc is 32 in.; the radius of the circle is 34 in.; find the arc.

(10) The chord of an arc is 2 ft.; the radius of the circle is 1 ft. 3 in.; find the arc.

(11) The chord of an arc is 64 ft.; the height of the arc is 24 ft.; find the arc.

(12) The chord of an arc greater than a semicircle is 70 ft.; the radius of the circle is 37 ft.; find the arc.

(13) The diameter of a circle is 106 ft.; find the lengths of the two arcs into which a chord of 90 ft. would divide it.

(14) The chord of an arc is 240 ft.; the height of the

arc is 64 ft.; find the diameter of the circle, and also the length of the arc.

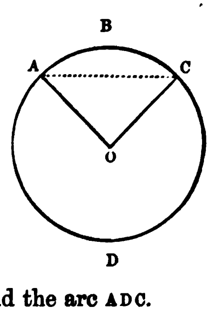
(15) The span of a bridge, the form of which is an arc of a circle, is 200 ft.; its height is 42 ft.; find the length of the arch of the bridge.

(16) The rise or height of a bridge, the form of which is an arc of a circle, is 12 ft., and its span is 78 ft.; find the length of the arch of the bridge.

(17) The span of each of the three arches of a bridge, the form of each being the arc of a circle, is 240 ft., and the height of the crown in each above the stone piers is 24 ft.; find the length of each arch.

XV. THE SECTOR OF A CIRCLE.

Definitions.—The sector of a circle is the figure bounded by two radii and the part of the circumference between them.



Thus AOCBA is the *sector* of a circle, being bounded by the two radii OA and OC, and the arc ABC between them; and is less than a semicircle.

AOC is the *angle* of the sector AOCBA, or the angle subtended by the arc ABC.

Also ADCOA is the *sector* of a circle greater than a semicircle, being bounded by the two radii OA and OC, and the arc ADC.

The straight line AC is the *chord* of the sector.

RULES.—(1) *To find the area of a sector, when the radius of the circle and angle of sector are given.*

Find the area of the circle by rules in Prob. XII.

Then we have the following proportion for finding the area of the sector :—

360° : number of degrees in angle of sector :: area of circle : required area of sector.

(2) *To find the area of a sector, when the radius of the circle and the arc are given.*

Multiply the length of the arc by the radius, and divide the product by 2.

Note 1.—To find the radius of the circle, when the area of the sector and the number of degrees in the angle of the sector are given.—The area of the circle must first be found from the following proportion :—

Number of degrees in the angle of the sector : 360° :: area of sector : required area of circle.

Then, having found the area of the circle, its radius can be found by Note 1, Prob. XII.

Note 2.—If the sector be greater than a semicircle, as, for instance, the sector ADCOA, then it will be better to find, first, the area of the sector ABCOA, which is less than a semicircle, by either of the preceding rules.

Then the sector ADCOA = area of circle − sector AOCBA.

Example 1.—Find the area of the sector of a circle whose radius is 10 inches, and whose arc subtends at the centre an angle of 18°.

Now, area of circle $= \frac{22}{7} \times 10^2 = \frac{2200}{7}$ sq. in.

Then, by Rule 1: 360° : 18° :: $\frac{2200}{7}$ sq. in. : required area of sector.

From this proportion, we find that $15\frac{5}{7}$ sq. in. is the area of the sector.

Example 2.—The radius of a circle is 4 ft.; the arc of a sector of the circle is 3 ft. 4 in.; find the area of the sector.

By Rule 2: Area of sector $=\frac{1}{2}\times$ arc $\times$ radius $=\frac{1}{2}\times 40\times 48$ $=960$ sq. in. $=6$ sq. ft. 96 sq. in.

Example 3.—The chord of a sector is 56 in.; the radius of the circle is 35 in.; find the area of the sector.

In this example, we must first find the chord of half the arc, which is 31·304 in.

Then, by Rule 3, Prob. XIV.: Length of the arc $=\frac{1}{3}(8\times 31{\cdot}304-56)=64{\cdot}81$ in.

Therefore, area of sector $=\frac{1}{2}$ (arc $\times$ radius) $=\frac{1}{2}\times 64{\cdot}81\times 35$ $=1134{\cdot}175$ sq. in.

Formulæ:
- I. Area of sector $=\frac{1}{2}\times$ arc $\times$ radius.
- II. 360° : number of degrees in angle of sector :: area of circle : required area of sector.

EXAMPLES.

(1) The radius of a circle is 14 in.; the angle which the arc subtends at the centre is 45°; find the area of the sector.

(2) The radius of the circle is 2 ft. 11 in.; the angle of the sector is 24°; find the area of the sector.

(3) The radius of a circle is 56 ft.; the angle subtended by the arc at the centre is $22\frac{1}{2}^\circ$; find the area of the sector.

(4) The area of the sector of a circle is 385 sq. ft.; the angle of the sector is 36°; find the radius of the circle, and also the whole perimeter of the sector.

(5) The area of a sector is 48·125 sq. ft.; the angle of the sector is 18°; find the radius.

(6) The chord of a sector is 30 ft.; the radius of the circle is 25 ft.; find the area of the sector.

(7) The chord of an arc is 40 ft.; the radius of the circle is 25 ft.; find the area of the sector.

(8) The arc of a circle greater than a semicircle is 86 ft.; the radius of the circle is 21 ft.; find the area of the sector.

(9) The chord of a sector is 20 ft.; the radius of the circle is 14·5 ft.; find the area of the sector.

(10) The chord of the sector of a circle is 56 ft.; t[illegible] radius of the circle is 53 ft.; find the area of the sector

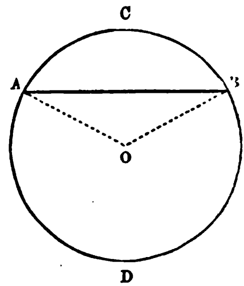
XVI. THE SEGMENT OF A CIRCLE.

Definitions.—The segment of a circle is any part of a circle bounded by an arc and its chord.

ABCA is a *segment* less than a semicircle, being bounded by the arc ACB and the chord AB.

ABDA is a *segment* greater than a semicircle, being bounded by the arc ADB and the chord AB.

N.B.—Segment ABCA = sector AOBCA − triangle AOB.



RULES.—(1) *To find the area of the segment of a circle.*

Find the area of the sector AOBCA, having the same arc as the segment, by Prob. XV., and subtract from it the area of the triangle AOB, formed by the radii and the chord.

(2) *To find the area of the segment of a circle, when the chord of the arc and its height are given.*

To $\frac{2}{3}$ of the product of the chord and height of the segment, add the cube of the height divided by twice the chord; the sum is the area of the segment, nearly.

Note 1.—To find the area of the segment ADBA, greater than a semicircle.—In this case, it will be better to find the area of the segment ABCA, less than a semicircle, by preceding rules.

Then, the area of the segment ADBA = area of circle − area of segment ABCA.

Example 1.—The radius of a circle is 35 ft., and the chord of a segment is 42 ft.; find the area of the segment.

Now, area of segment = sector AOBCA − triangle AOB.

By Prob. XIII., the chord of half the arc = 22·13 ft.; and by Prob. XIV., the arc = 45·01 ft., nearly.

Therefore, area of sector $= \frac{1}{2}(45 \cdot 01 \times 35) = 787 \cdot 67$ sq. ft.

And area of triangle, having its sides 35 ft., 35 ft., and 42 ft respectively, is 588 sq. ft.

Therefore, area of segment $= 787 \cdot 67 - 588 = 199 \cdot 67$ sq. ft., nearly.

Example 2.—Find the area of the segment of a circle whose chord is 35 ft., and the height of the segment is 9·6 ft.

By Rule 2: $\text{Area} = \frac{2}{3}(35 \times 9 \cdot 6) + \frac{9 \cdot 6^3}{35 \times 2} = \frac{2}{3} \times 336 + \frac{884 \cdot 736}{70}$

$= 224 + 12 \cdot 639 = 236 \cdot 639$ sq. ft., nearly.

$$\text{Formulæ} \begin{cases} \text{I. Segment} = \text{sector} - \text{triangle.} \\ \text{II. Segment} = \frac{2}{3}(\text{chord} \times \text{height}) + \dfrac{\text{height}^3}{2\ \text{chord}} \end{cases}$$

Examples.

[Work Examples 1 and 2 by Rule 1; the rest by Rule 2.]

(1) Find the area of a segment, when the chord of the arc is 56 ft. and the radius of the circle is 35 ft.

(2) The chord of a segment is 30 ft., and the radius of the circle is 17 ft.; find the area of the segment.

(3) The chord of a segment is 24 ft., and its height is 9 ft.; find the area of the segment.

(4) The chord of a segment is 40 ft., and its height is 15 ft.; find the area of the segment.

(5) The chord of a segment greater than a semicircle is 24 ft.; the diameter of the circle is 26 ft.; find the area of the segment.

(6) A room, which is 20 ft. long and 15 ft. wide, has an opening or recess at one end, in the shape of a segment of a circle, the opening or chord of the recess being 15 ft. and its greatest width 4 ft.; find the area of the whole room.

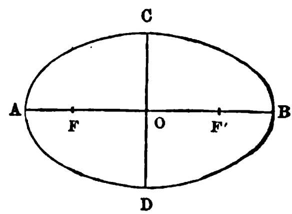
(7) An arched gateway is 30 ft. wide, and measures 20 ft. from the ground to the spring of the arch; and the distance also from the ground to the crown of the arch is 30 ft. Find how many square feet of timber will be required for blocking it up.

(8) Find how many sq. ft. of brickwork are used in blocking up one of the arches in a railway viaduct:

The span of the arch is 60 ft., the height above the piers is 20 ft., and the distance from the ground to the spring of the arch is 20 ft.

XVII. THE ELLIPSE.

Definitions.—An ellipse, in simple language, may be defined to be a plane figure bounded by a curved line, called the circumference, and which has a longer and a shorter diameter.



The longer diameter AB is called the *transverse diameter*, or *major axis*.

The shorter diameter CD is called the *conjugate diameter*, or *minor axis*.

F and F′ are the foci of the ellipse.

RULES.—(1) *To find the circumference or perimeter of the ellipse.*

Multiply half the sum of the two diameters by $\frac{22}{7}$.

(2) *To find the area of an ellipse.*

Multiply the product of the two diameters by $\frac{11}{14}$.

Or, multiply the product of half the diameters by $\frac{22}{7}$.

Note 1.—To find either the longer or the shorter diameter, when the circumference and the other diameter are given.—Subtract the given diameter from $\frac{7}{11}$ of the circumference.

Note 2.—To find either diameter, when the area and the other diameter are given.—Divide $\frac{14}{11}$ of the area by the given diameter.

Note 3.—An ellipse is equal to a circle whose diameter is a mean proportional between the two diameters; that is, the area of the ellipse ACBD is equal to the area of a circle whose diameter is $\frac{AB + CD}{2}$.

Or, in other words, the ellipse is a mean proportional between its inscribed and circumscribed circles; that is, between the circles about the transverse and conjugate diameters. For instance, if a circle were described having AB for its diameter, and another circle were described having CD for its diameter, then the area of the ellipse is equal to half the sum of the areas of these two circles.

Note 4.—By the rules given above, the circumference and the area of an ellipse will be found with sufficient accuracy for all practical purposes. If, however, greater accuracy is desirable, then, in Rule 1, substitute 3·1416 as multiplier, instead of $\frac{22}{7}$; and in Rule 2, ·7854 for $\frac{11}{14}$.

Note 5.—The area of an elliptical ring is the *difference* between the areas of the *outer* and *inner* ellipse.

Example 1.—Find the perimeter or circumference of a

ellipse whose transverse diameter is 28 ft. 6 in., and conjugate diameter is 20 ft. 6 in.

By Rule 1: Circumference$=\frac{1}{2}$ sum of diameters $\times \frac{22}{7}=\frac{1}{2}$ (28 ft. 6 in.$+$20 ft. 6 in.) $\times \frac{22}{7}=\frac{1}{2}\times 49$ ft. $\times \frac{22}{7}=77$ ft.

Example 2.—Find the area of an ellipse, whose major axis is 156 ft., and minor axis is 120 ft.

By Rule 2: Area$=156\times 120\times \frac{11}{14}=18480$ sq. ft.

Example 3.—The perimeter of an ellipse is 28 yds. 1 ft. 3 in., and its minor axis is 22 ft. 9 in.; find the major axis.

Now, 28 yds. 1 ft. 3 in.$=$1023 in.; and 22 ft. 9 in.$=$273 in.
By Note 1:
The major axis$=\frac{7}{11}$ of the circumference$-$minor axis
$=(\frac{7}{11}$ of 1023$)-273=651-273=378$ in.
$=31$ ft. 6 in.

Example 4.—The area of an ellipse is 28 sq. ft. 60 sq. in., and its transverse diameter is 7 ft. 9 in.; find the conjugate diameter.

Now, 28 sq. ft. 60 sq. in.$=$4092 sq. in.; and 7 ft. 9 in.$=$93 in.
By Note 2:
The conjugate diameter$=\frac{14}{11}$ of the area$\div$given diameter
$=\frac{14}{11}$ of $4092\div 93=5208\div 93=56$ in.
$=4$ ft. 8 in.

Formulæ:
- I. Circumference$=\frac{1}{2}$ (sum of the two diameters) $\times \frac{22}{7}$.
- II. Area$=$transverse diameter $\times$ conjugate diameter $\times \frac{11}{14}$.
- III. Required diameter$=\frac{7}{11}$ of the circumference$-$given diameter.
- IV. Required diameter$=\frac{14}{11}$ of the area $\div$given diameter.

EXAMPLES.

Find the circumference of the ellipse whose transverse and conjugate diameters are, respectively—

(1) 48 in., and 36 in. (2) 13 ft., and 8 ft.
(3) 87 yds. 1 ft., and 34 yds.
(4) 3 ch. 80 lks., and 3 ch. 20 lks.

Find the minor axis of an ellipse, when its circumference and its major axis are, respectively—

(5) Circumference 286 ft., and major axis 121 ft.
(6) Circumference 385 ft., and major axis 145 ft.
(7) Circumference 47 ft. 8 in., and major axis 16 ft. 8 in.
(8) Circumference $\frac{1}{4}$ mile, and major axis 165 yds.

Find the area of the ellipse, when its major and minor axes are, respectively—

(9) 110 ft., and 98 ft. (10) 40 ft. 10 in., and 29 ft. 2 in.
(11) 42 yds., and 32 yds. 2 ft.
(12) 102 yds. 2 ft., and 93 yds. 1 ft.
(13) 2 ch. 38 lks., and 2 ch.

Find the conjugate diameter of the ellipse, when its area and its transverse diameter are, respectively—

(14) Area 550 sq. ft., and transverse diameter 35 ft.

(15) Area 15 sq. ft. 40 sq. in., and transverse diameter 5 ft. 10 in.

(16) Area 7 ac. 0 rood 20 poles 11 yds., and transverse diameter 224 yds.

(17) Area 2 roods 8 poles, and transverse diameter 2 ch. 80 lks.

(18) In a rectangular plot of land, which is 100 yds. long and 70 yds. broad, is dug a fish-pond, in the shape of an ellipse, whose greater diameter is 98 yds. and less diameter is 60 yds.; the remaining part is to be gravelled, at 3*d.* per sq. yd. Find the expense.

(19) The rental of a field, in the shape of an ellipse, whose greater diameter is 140 yds., at £6 1*s.* per acre, is £15 8*s.*; find the shorter diameter.

(20) The building of a wall round an elliptical plot of land, whose greater diameter is 106 yds., at 2*s.* 9*d.* per yard, costs £36 6*s.*; find the length of the shorter diameter.

(21) A lawn, in the shape of an ellipse, whose greater and less diameter are, respectively, 98 ft. and 58 ft., is surrounded by a walk 1 yd. wide; find the cost of gravelling it, at 6*d.* per sq. yd.

SOLIDS.

Definitions.

1. The *volume*, *solidity*, or *solid content* of a solid is the space (cubic yds., ft., or in.) that it occupies.

2. The *surface* of a solid is its outside.

3. The different solids have been so fully explained in the following chapters, that it has not been thought necessary to give any definition of them here. The Student must try to understand the explanations given of each solid before attempting the Examples.

4. Tables:

1728 cubic in. = 1 cubic ft.
27 cubic ft. = 1 cubic yd.

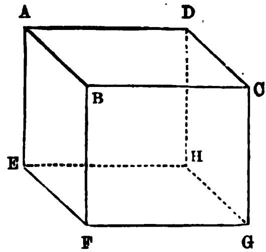
1 cubic ft. of water weighs 1000 oz. avoir., very nearly.

1 cubic ft. contains $6\frac{1}{4}$ gallons, very nearly.

1 gallon contains 277 cubic in., or, more nearly, 277·274 cubic in., or $277\frac{1}{4}$ cubic in.

			avoirdupois		
A cubic in.	of gold	weighs	11·194 oz., or	$11\frac{1}{5}$ oz.,	nearly.
„	lead	„	6·604 oz., or	$6\frac{3}{5}$ oz.	„
„	silver	„	6·059 oz., or	6 oz.	„
„	copper	„	5·106 oz., or	$5\frac{1}{10}$ oz.	„
„	iron	„	4·526 oz., or	$4\frac{1}{2}$ oz.	„

XVIII. THE CUBE.



Definition.—A cube is a solid bounded by six equal square faces. Each of the surfaces ABCD, BFGC, &c., is called a *face* or *side*. The straight lines which bound these square faces—as, for instance, the straight lines AB, BF, FG, &c.—are called *edges*, but *generally sides.*

N.B.—By the *side* of a cube is usually meant not one of the *side faces*, as ABCD, but simply one of its edges, as AB, BF, FG, &c. The Student will easily see whether the expression *side* refers to the *length* of an *edge*, or to the *area* of one of the *faces* of the cube.

RULES.—(1) *To find the volume of a cube.*

Cube the side (edge) given.

(2) *To find the whole surface of a cube.*

Multiply the square of the side (edge) given by 6.

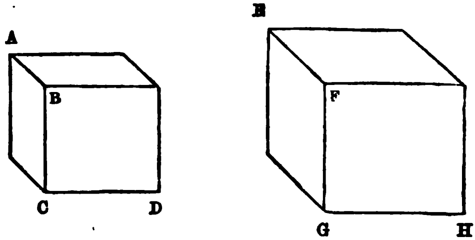
Note 1.—To find the side (edge) of a cube, when its volume is given.—Extract the cube root of the volume.

Note 2.—To find the side (edge) of a cube, whose volume shall bear a certain proportion to that of a given cube, we may proceed thus:

(*a*) Supposing AB, the side or edge of the cube ABCD is given, and it is required to find EF, the side or edge of another cube EFGH, whose volume is double that of the cube ABCD, we have

$$EF^3 = 2 \times AB^3.$$

(Since the volume of cube EFGH is double that of cube ABCD; and the volume of EFGH is EF^3, and also that of ABCD is AB^3.)



So also the cube roots of these quantities are equal; that is,

$$EF = \sqrt[3]{2 \times AB^3} = AB \times \sqrt[3]{2}.$$

Hence the side EF is found by multiplying AB, the side or edge of the given cube, by $\sqrt[3]{2}$.

If the volume of the cube EFGH is treble that of the cube ABCD, then the side EF is found by multiplying AB by $\sqrt[3]{3}$. If four times as much, then the multiplier is $\sqrt[3]{4}$; and so on.

But if the volume of the required cube is half that of the given cube, we shall, in that case, find its side or edge by *dividing* the side of the given cube by $\sqrt[3]{2}$; if it is to be one-third, then its side is found by *dividing* by $\sqrt[3]{3}$; and so on.

(*b*) The following method, however, will be the more readily understood by beginners.

If the volume of the required cube is double that of the given cube, then, multiply the volume of the *given* cube by 2, and the product is the volume of the *required* cube. Having thus found the volume of the required cube, its side or edge can be found by Note 1.

If the volume is three times as much, then, multiply the volume of the given cube by 3, and the cube root of this product gives the side or edge of the required cube.

But if the volume is half as much, then, divide the volume of the given cube by 2; and the cube root of the quotient is the side or edge of the required cube.

The same remarks will apply to the case when its volume is one-third, one-fourth, &c.

[To save trouble, the following cube roots are given: $\sqrt[3]{2}=1{\cdot}259+$; $\sqrt[3]{3}=1{\cdot}442$; $\sqrt[3]{4}=1{\cdot}587$, &c.]

Example 1.—Find the volume of a cube which measures 3 ft. 6 in. every way.

Volume $=$ cube of side given $=(3\frac{1}{2})^3=\frac{7}{2}\times\frac{7}{2}\times\frac{7}{2}=\frac{343}{8}$ cubic ft. $=42\frac{7}{8}$ cubic ft.

Example 2.—The volume of a cube is 91·125 cubic ft.; find the length of one of its sides.

Length of each side $=\sqrt[3]{\text{volume}}=\sqrt[3]{91{\cdot}125}=4{\cdot}5$ ft. $=$ 4 ft. 6 in.

Example 3.—The side of a cubical vessel is 10 ft.; find the side of another cubical vessel whose volume is eight times that of the former.

Now, by Note 3, *a*:

The side of the required cube $=$ side of the given cube $\times\sqrt[3]{8}$

$$=10\times2=20 \text{ ft.}$$

Hence, a cubical vessel whose side is 20 ft. contains eight times the volume of a cube whose side measures only 10 ft.

$$\text{Formulæ}\begin{cases}\text{I. Volume}=\text{side}^3.\\ \text{II. Whole surface}=\text{area of side face}\times 6.\\ \text{III. Side or edge}=\sqrt[3]{\text{volume}}.\end{cases}$$

Examples.

[For the sake of uniformity in the answers, the volumes are given in cubic feet and inches; and the surfaces in square feet and inches.]

Find the volume, and also the *whole* surface, of the following cubes, whose sides or edges are, respectively—

(1) Side 3 in. (2) Side 1 ft. 2 in.
(3) Side 8·5 in. (4) Side 2 ft. 3 in.
(5) Side 8 ft. 6 in. (6) Side 20 ft.
(7) Side 2 yds. 1 ft. 3 in. (8) Side 5 yds. 2 ft. 6 in.
(9) 6 yds. 1 ft. 4 in.

Find the length of each side of the cube whose volume is—

(10) Volume 343 cub. in.
(11) Volume 2 cub. ft. 1457 cub. in.
(12) Volume 15 cub. ft. 1080 cub. in.
(13) Volume 68 cub. ft. 145 cub. in.
(14) Volume 181 cub. yds. 26 cub. ft.
(15) Volume 21 cub. yds. 25·704 cub. ft.

(16) A cubic foot of water weighs 1000 oz. avoirdupois; find the weight of the water in a cubical cistern, each of whose sides measures 6 ft.

(17) How much lead is used in lining the sides and bottom of a cubical vessel that contains 729 cubic ft. of water?

(18) A cubical vessel measures 5 ft. every way; find the length of another vessel, of the same shape, that will hold twice the quantity of water.

(19) The gallon contains 277·274 cub. in. of water; find the length of a cubical cistern that holds 1000 gallons.

(20) The weight of a cubic foot of water is 1000 oz. avoirdupois, and marble is 2·7 times heavier than water; find the weight of a cubical block of marble whose side ...ures 5 ft.

(21) A cubic foot of water weighs 1000 oz. avoirdupois; find the side of a cube made of gold that weighs the same, when gold is 19·3 times heavier than water.

(22) If a cubical vessel requires 320 sq. ft. of lead for lining its sides and bottom, find the number of cubic feet of water that it contains.

(23) A plumber, being asked to make a cubical cistern to contain 216 cub. ft. of water, is afterwards told to make it hold half as much more, without altering its shape. Thinking that this will be accomplished by making its dimensions half as much more than he intended at the first, he accordingly makes each side measure 9 ft. Find what *ought* to have been the length; and also *how much too large* was the cistern that he made.

(24) The carpeting of a room, in the shape of a cube, with carpet 24 in. wide, at 3*s.* 4*d.* per yd., costs £6 5*s.*; find the number of cubic feet of air that the room contains.

(25*) A cubical box with a lid is covered with sheet-lead which weighs 4 lbs. per sq. ft., and 294 lbs. of lead are used; what is the side of the box?

(26) A cubical vessel holds 421 cub. ft. 1512 cub. in. of water; find how many sq. ft. of lead will be required for lining its sides and bottom.

(27) The cost of polishing a cubical block of Cornish granite, at 3*s.* 4*d.* per sq. ft., is £36; find its weight, supposing that 1 cub. ft. of water weighs 1000 oz., and that granite is 2·7 times heavier than water.

(28) A cubical vessel is constructed to hold 775 gallons of water; find its length, allowing 6¼ gallons to the cubic foot.

(29) A cubic foot of water weighs 1000 oz. avoirdupois, and a cubical vessel is to be made which shall hold 10 cwt. of water; find what ought to be its length.

XIX. THE RECTANGULAR PARALLELOPIPED.

Definitions.—The rectangular parallelopiped is a solid contained by six rectangular faces, each of them being equal and parallel to its opposite face.

The annexed figure represents a rectangular parallelopiped. All the faces bounding this figure are rectangles, every opposite two of which are equal and

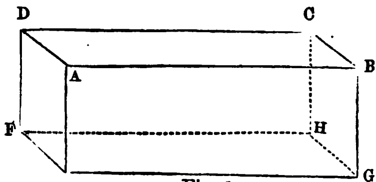


Fig. 1.

parallel; thus the face ABCD is equal and parallel to the face EFGH, and the face ADFE to the face BCGH; and so on.

EFGH is also called the *base*; and the faces AEFD, BGHC are frequently called *ends*.

But base, ends, &c. are all included under the general term *faces*.

Also, AB or EG is called the *length*; AD or EF the *breadth*; and AE or DF the *height* or *depth*.

The *whole surface* consists of the areas of the two ends AEFD, BCHG, and four *side faces* AEGB, DFHC, ABCD, EFGH.

Rules.—(1) *To find the volume of a rectangular parallelopiped.*

Multiply the length by the breadth, and this product by the height.

Or, multiply the area of the base by the height.

(2) *To find the whole surface.*

Multiply the perimeter of the end (which is $=2$ width $+2$ depth) by the length, and the product is the area of the side faces; to which add

twice the area of one end, and the sum is the whole surface.

Or, add together the areas of the two ends and four sides, and the sum is the *whole surface.*

Note 1.—To find one dimension, either the length, breadth, or depth, when the volume and the other dimensions are given.—Divide the volume by the product of the two given dimensions, and the quotient is the required dimension.

Note 2.—A parallelopiped may *also* be *rhomboidal* or *oblique* (fig. 2); in which case the edges are *not* perpendicular to each other—that is, all the faces are not rectangles.—The annexed figure represents an oblique parallelopiped, in which some of the side faces, as ABCD, EFGH, &c. are rectangles, but others, as AEGB, DFHC, &c. are rhomboids.

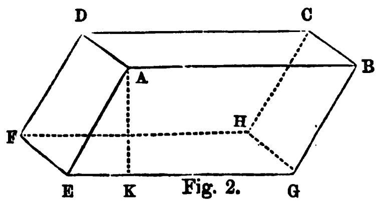


Fig. 2.

The volume of an oblique parallelopiped may be found thus:—Multiply the length AB by the breadth AD, and this product by the *perpendicular* height AK (and not by AE).

Note 3.—To increase or diminish *proportionally* the dimensions of a rectangular parallelopiped, so as to effect a corresponding increase or diminution of the volume.

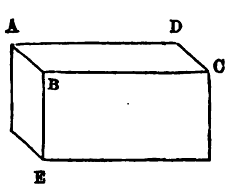


Fig. 3.

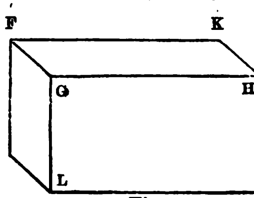


Fig. 4.

Supposing it is required to find the dimensions of the parallelopiped (fig. 4) whose volume is twice that of the given parallelopiped (fig. 3), we may proceed thus: Since the

volume FGKL is twice the volume ABED, and that the volume of FGKL=GH×FG×GL, and also the volume of ABED=BC×AB×BE, we have,

$$GH \times FG \times GL = 2 \times BC \times AB \times BE;$$

that is, length × breadth × depth (of the required figure) = length $\sqrt[3]{2}$ × breadth $\sqrt[3]{2}$ × depth $\sqrt[3]{2}$ (of given figure).

Hence, if the volume is to be twice as large, then multiply each given dimension by $\sqrt[3]{2}$.

If the volume is to be three times as large, then multiply each given dimension by $\sqrt[3]{3}$.

If the volume is to be four times as large, then multiply each given dimension by $\sqrt[3]{4}$; and so on.

Note 4.—To find the cubic feet or cubic inches of the *material* used in the construction of a rectangular cistern or box.

First find the volume of a rectangular parallelopiped whose length, breadth, and depth are the same as the *external* length, breadth, and depth of the cistern or box; then find the volume of another rectangular parallelopiped whose length, breadth, and depth are the *internal* length, breadth, and depth of the cistern or box.

Then the difference of these two volumes gives the number of cubic feet or cubic inches of the material.

N.B.—If the material employed is *uniformly* thick,

then: *external* length = *internal* length + 2 thickness of material.

external breadth = *internal* breadth + 2 thickness of material.

external depth = *internal* depth + thickness of material (if the vessel is *without* a lid).

= *internal* depth + 2 thickness of material (if the vessel has a lid).

Example 1.—Find the volume of a rectangular parallelopiped whose length is 6 ft., breadth 4 ft. 6 in., and height 3 ft. 6 in.

The volume$=6$ ft. $\times 4\frac{1}{2}$ ft. $\times 3\frac{1}{2}$ ft.$=6\times\frac{9}{2}\times\frac{7}{2}=\frac{189}{2}$ cub. ft. $=94\frac{1}{2}$ cub. ft.$=94$ cub. ft. 864 cub. in.

Example 2.—A rectangular cistern is 24 ft. long and 15 feet broad; find its depth that it may hold 4320 cub. ft. of water.

By Note 1:

The depth=volume divided by the product of the two given dimensions.

$=4320\div(24\times15)=4320\div360=12$ ft.

Example 3.—A box without a lid, in the shape of a rectangular parallelopiped, is made of wood 1 in. thick; externally the length is 3 ft. 6 in., the breadth 3 ft., and the depth 2 ft. Find the number of cubic inches of wood used in its construction.

The *external* dimensions are 42 in., 36 in., and 24 in.; therefore the volume of the rectangular parallelopiped having these dimensions is 36288 cub. in.

Again, the *internal* dimensions are $(42-2)$ in., $(36-2)$ in., and $(24-1)$ in., or 40, 34 and 23 in.; therefore the volume of the rectangular parallelopiped having these dimensions is 31280 cub. in.

Hence, volume of wood$=36288-31280=5008$ cub. in. $=2$ cub. ft. 1552 cub. in.

Formulæ:

I. Volume=length × breadth × height.

II. Whole surface= {2(breadth + height) × length} + 2(breadth × height).

Or, sum of the areas of the six rectangular faces.

III. Required dimension

$$=\frac{\text{volume}}{\text{product of 2 given dimensions.}}$$

Examples.

Find the volume, and also the whole surface, of rectangular parallelopipeds with the following dimensions :—

(1) Length 10 in., breadth 8 in., and depth 6 in.

(2) Length 2 ft. 4 in., breadth 2 ft., and height 1 ft. 8 in.

(3) Length 4 ft. 8 in., breadth 3 ft. 10 in., and height 3 ft.

(4) Length 6 yds. 2 ft., breadth 5 yds. 1 ft., and height 2 yds. 2 ft.

(5) Length 5 yds. 1 ft. 6 in., breadth 4 yds. 2 ft. 3 in., and height 3 yds. 2 ft. 6 in.

(6) Length 6·75 ft., breadth 5·5 ft., and height 4·25 ft.

Find the volume of the following rectangular parallelopipeds, when the area of the base and the height are, respectively—

(7) Area of base 97 sq. in., and height 10 in.

(8) Area of base 3 sq. ft. 20 sq. in., and height 1 ft. 3 in.

(9) Area of base 11 sq. ft. 40 sq. in., and height 5 ft. 3 in.

Find the height or depth of a rectangular parallelopiped, when its volume, length, and breadth are, respectively—

(10) Volume 378 cub. in., length 9 in., and breadth 7 in.

(11) Volume 1680 cub. in., length 1 ft. 2 in., and breadth 1 ft.

(12) Volume 3 cub. ft. 864 cub. in., length 2 ft., and breadth 1 ft. 6 in.

(13) Volume 1164 cub. ft. 1008 cub. in., length 12 ft. 6 in., and breadth 10 ft. 9 in.

(14) If the area of the bottom of a box is 27 sq. ft. 72 sq. in.; find what must be its depth, so that the volume of the box may be 123 cub. ft. 1296 cub. in.

(15) Find the price of a balk of timber 39 ft. 6 in. long, and 3 ft. 7 in. thick each way, at 2*s*. 6*d*. a cub. ft.

(16) A tank is 30 ft. 9 in. long, 16 ft. 7 in. wide, and 6 ft. 4 in. deep; find how much water it will hold in cubic feet and inches.

(17) A cellar, which measures 12 ft. long and 9 ft. wide, is flooded to a depth of 4 in.; find the weight of the water, supposing that 1 cubic foot of water weighs 1000 oz. avoirdupois.

(18**) What weight of water will a rectangular cistern contain, the length being 4 ft., the breadth 2 ft. 6 in., and the depth 3 ft. 3 in., when a cubic foot of water weighs 1000 oz. avoirdupois?

(19*) How much lead will be required to line a cistern, open at the top, which is 4 ft. 6 in. long, 2 ft. 8 in. wide, and contains 42 cubic feet?

(20*) How many bricks will be required to build a wall 90 ft. long, 18 in. thick, and 8 ft. high; a brick being 9 in. long, $4\frac{1}{2}$ in. wide, and 3 in. deep?

(21) A quantity of earth, in the shape of a rectangular parallelopiped, which measures 6 ft. long, 5 ft. wide, and 4 ft. deep, is spread uniformly over a rectangular court which measures 48 ft. long, and 10 ft. wide; find the depth of the earth.

(22) A box is 4 ft. 6 in. long, 3 ft. 6 in. broad, and 2 ft. 6 in. deep; how many books will it contain, when each book measures 7 in. long, 6 in. wide, and $1\frac{1}{2}$ in. thick?

(23) A book has the following dimensions: 8 in. long, 6 in. wide, and $1\frac{1}{4}$ in. thick; find the depth of the box, whose length and breadth are 3 ft. 4 in. and 2 ft. 6 in., that it may contain 400 such books.

(24) The water in a large rectangular cistern, which is 15 ft. 6 in. long and 12 ft. wide, has sunk 3 in.; find how many cubic feet of water have been drawn off.

(25) How many bricks will be required for a wall 25 yds. long, 15 ft. high, and 1 ft. $10\frac{1}{2}$ in. thick; each brick being 9 in. long, $4\frac{1}{2}$ in. wide, and 3 in. deep?

(26) The internal dimensions of an open cistern, made of iron 2 in. thick, in the shape of a rectangular parallelopiped, are length 4 ft. 6 in., breadth 3 ft., and depth 2 ft. 6in.;

find how many cubic feet of iron were used in its construction.

(27) Find the expense of covering a flat roof, 17 ft. 4 in. long and 13 ft. 4 in. wide, with sheet-lead $\frac{1}{16}$ of an inch thick, supposing that a cubic inch of lead weighs $6\frac{1}{2}$ oz. avoirdupois, and that 1 lb. costs $3\frac{1}{2}d.$

(28) If a cubic foot of marble weighs 2·716 times as much as a cubic foot of water, find the weight of a block of marble 9 ft. 6 in. long, 2 ft. 3 in. broad, and 2 ft. thick, supposing a cubic foot of water weighs 1000 oz. avoirdupois.

(29*) The beams of wood used in building a house are 3 in. thick and 10 in. wide: 200 of them are used, which together amount to 1000 cub. ft. What is the length of each beam?

(30) The length and breadth of a rectangular cistern are 10 ft. and 5 ft.; what must be its depth that it may contain 1000 gallons of water, supposing that a gallon contains 277·274 cub. in.?

(31) An open rectangular cistern is made of cast iron 1 in. thick, and has for its external length, breadth, and depth 5 ft., 4 ft., and 3 ft. respectively; find its weight when empty, when iron weighs 7·2 times as much as water, and 1 cubic foot of water weighs 1000 oz. avoirdupois.

(32**) How many gallons of water will a cistern hold whose length, breadth, and depth are 5 ft. 6 in., 3 ft. 9 in., and 1 ft. 3 in.? What would be the weight of water contained (a gallon contains 277 cub. in., and a cub. ft. of water weighs 1000 oz. avoirdupois)?

(33) An open cistern, which is 6 ft. long and $4\frac{1}{2}$ ft. wide, contains 108 cub. ft. of water; find how many cubic feet of lead will be required for lining its sides and bottom, when the lead is $\frac{1}{8}$ in. thick.

(34) If a cubic foot of gold may be made to cover uniformly and perfectly 432000000 sq. in., find the thickness of the coating of gold.

(35) Rain has fallen to the depth of half an inch; find how many cubic feet of water have fallen on an acre of land.

(36) The length, breadth, and depth of a rectangular cistern are 8 ft., 6 ft., and 4 ft.; find the length, breadth, and depth (proportional to the former) of another rectangular cistern that shall contain eight times the quantity of water.

(37) The length of a rectangular cistern is 10 ft., its breadth is 8 ft., and its depth is 6 ft.; find the length, breadth, and depth (proportional to the former) of another cistern that shall contain only half the quantity of water.

(38) The water in a large rectangular cistern, whose length is 12 ft. and breadth 10 ft., has sunk 8 in. on account of a leakage; find how many gallons will have been wasted, when there are $6\frac{1}{4}$ gallons in a cubic foot.

XX. THE RIGHT-PRISM AND THE RIGHT-CYLINDER.

Definitions.—I. A prism is a solid whose two ends ABC, DEF are similar, equal, and parallel plane figures, and whose side faces ABED, BCFE, &c. are parallelograms.

The ends of a prism may be triangles, as fig. 1; trapezoids, or pentagons, as in fig. 2; hence a prism is called a triangular prism, pentagonal prism, &c., according to the figure that it has for its ends.

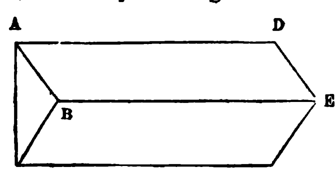


Fig. 1.

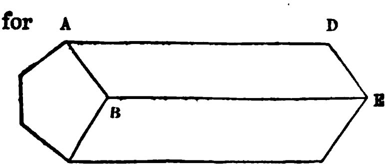


Fig. 2.

The line drawn from the *centre* of *one* end to the *centre* of the *other* end is called the *axis* of the prism.

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A *right-prism* has the axis at *right angles* to the ends of the prism; or it may be defined to be a prism that has all its side faces rectangles.

In an *oblique prism* the axis is *not at right angles* to the ends, and its side faces are not rectangles.

The *whole surface* of a prism will consist of the areas of the several side faces ABED, BCFE, &c., and the areas of the two ends ABC and DEF.

II. A cylinder is a solid having its ends two equal and parallel circles.

The straight line CE between the two ends, from centre to centre, is called the *axis* of the cylinder.

In the right-cylinder (fig. 3) this axis CE is *perpendicular* to the ends.

In the oblique cylinder (fig. 4) the axis CE is *not perpendicular* to the ends.

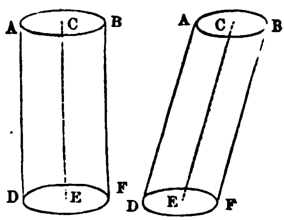


Fig. 3. Fig. 4.

The *whole surface* of a cylinder will consist of the areas of the two circular ends (ABC and DEF) and the curved or round surface.

The *height*, or *altitude*, or *length* of a cylinder or prism is the *perpendicular* drawn from one end to any point in the other end; and in the case of a right-prism or right-cylinder, the height or length of the figure is the same as the axis.

N.B.—The cylinders and the prisms mentioned in the following Examples are right-cylinders and right-prisms.

RULES.—(1) *To find the volume of a right-cylinder or a right-prism.*

Multiply the area of the base by the height or length.

(2) *To find the curved surface of a right-cylinder, or the side faces of a right-prism.*

Multiply the circumference or perimeter of the end by the height or length.

(3) *To find the whole surface of a right-cylinder or right-prism.*

To the curved surface of the cylinder, or to the area of the side faces of the prism (found by Rule 2), add the areas of the two ends.

Note 1.—To find the length or height of a right-cylinder or right-prism.—Divide the volume by the area of the base or end.

Note 2.—To find the area of the base or end of a right-cylinder or right-prism.—Divide the volume by the height or length.

Note 3.—The end of a prism being any plane figure, its area, of course, will have to be found by the rules given for finding the area of that particular figure in the Mensuration of Superficies.

Note 4.—The volume of an oblique cylinder or oblique prism will be the same as that of a right-cylinder or right-prism having the *same area for the base*, and lying between the *same parallel planes*.

Thus, the volume of the oblique cylinder ABDE (fig. 5) is equal to the volume of the right-cylinder GDEH, if they have the same area DE for the base, and lie between the same parallel planes.

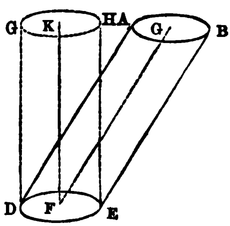


Fig. 5.

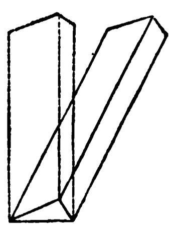


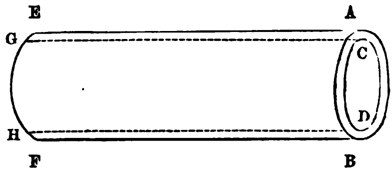
Fig. 6.

Hence, the volume of an oblique cylinder ABDE will be found by multiplying the area of the circular base DE, *not* by the length of the cylinder, but by the *perpendicular* FK.

But with regard to the curved surface of an oblique cylinder, no simple expression can be given for finding it.

The same remarks will apply to the case of an oblique prism.

Note 5.—To find the number of cubic feet or cubic inches in a cylindrical shell or pipe.



(*a*) Find the volume of the cylinder ABEF, and then of the hollow cylindrical part CDGH. Then the number of cubic feet in the pipe = difference between the volumes of these two cylinders.

(*b*) Or, take a section ACDB of the pipe perpendicular to the axis. Find the area of the ring by Note 9, Prob. XII., and then multiply the area of the ring by the length of the pipe; and the product is the volume of the pipe.

Observe: The *outer* diameter AB = the *inner* diameter + twice thickness of pipe; and *inner* diameter CD = *outer* diameter AB − twice thickness of pipe.

Note 6.—(*a*) Given the diameter and the length of a right-cylinder; to find the diameter and the length (proportional to the former) of *another* cylinder whose volume shall be so many times, more or less, than that of the *given* cylinder.

Following the reasoning given in Note 3, Prob. XIX., it will be found that the length and the diameter of the *required* cylinder can be obtained by multiplying or dividing (as the case may be) the length and the diameter of the *given* cylinder by $\sqrt[3]{}$ (number of times the required volume is greater or less than the given volume).

Thus, for instance, if the volume of the required cylinder

is twice that of the given cylinder, and a proportional increase has to be made in *both* the length and diameter,

then the diameter of *required* cylinder = diameter of *given* cylinder $\times \sqrt[3]{2}$;

and length of *required* cylinder = length of *given* cylinder $\times \sqrt[3]{2}$.

(*b*) But, on the other hand, if the increase is to be effected *only* in the *diameter*, whilst the *length* remains the *same*, then the diameter of *required* cylinder = diameter of *given* cylinder $\times \sqrt[2]{}$ (number of times the required volume exceeds the given volume).

(*c*) If an increase is to be effected *only* in the *length*, whilst the diameter remains the *same*, then the length of the *required* cylinder = length of *given* cylinder $\times$ number of times the required volume exceeds the given volume.

Example 1.—Find the volume of a cylinder, the diameter of whose base is 14 in., and whose length is 3 ft.

Area of base $= 14^2 \times \frac{11}{14} = 154$ sq. in.

Then the volume $= 154 \times 36 = 5544$ cub. in. $= 3$ cub. ft. 360 cub. in.

Example 2.—Find the curved surface, and the whole surface of a cylinder, whose length is 6 ft., and the radius of its end is 10 in.

The circumference of base $= 20$ in. $\times \frac{22}{7} = \frac{440}{7}$ in.

Then the curved surface $= \frac{440}{7} \times 72 = 4525\frac{5}{7}$ sq. in. $= 31$ sq. ft. $61\frac{5}{7}$ sq. in.—1st. Answer.

Also the areas of the two ends $= 2 \times 10^2 \times \frac{22}{7} = \frac{4400}{7}$ sq. in. $= 628\frac{4}{7}$ sq. in. $= 4$ sq. ft. $52\frac{4}{7}$ sq. in.

Then the *whole surface* = curved surface + areas of two ends

$= 31$ sq. ft. $61\frac{5}{7}$ sq. in. $+ 4$ sq. ft. $52\frac{4}{7}$ sq. in.

$= 35$ sq. ft. $114\frac{2}{7}$ sq. in.—2nd Answer.

Example 3.—Find the number of cubic inches in a cast-iron pipe, whose length is 10 ft., and the outer and inner diameters of its ends are 7 in. and 6 in. respectively.

By Note 9, Prob. XII.: Area of circular ring at each end $=\overline{7+6}\times\overline{7-6}\times\frac{11}{14}$.

$=13\times 1\times\frac{11}{14}=\frac{143}{14}$ sq. in.

Then, volume of pipe $=\frac{143}{14}\times 120=\frac{8580}{7}$ cub. in. $=1225\frac{5}{7}$ cub. in.

Example 4.—A cylindrical iron rod, $2\frac{1}{2}$ in. thick, is made out of 2 cub. ft. of iron; find its length.

Area of the end of the rod $=(\frac{5}{2})^2\times\frac{11}{14}=\frac{25}{4}\times\frac{11}{14}=\frac{275}{56}$ sq. in.

Then, length of the rod = volume ÷ area of end

$=2$ cub. ft. $\div\frac{275}{56}$ sq. in. $=3456$ cub. in. $\div\frac{275}{56}$ sq. in.

$=703\frac{211}{275}$ in. $=58$ ft. $7\frac{211}{275}$ in.

Formulæ:
- I. Volume = area of base × height.
- II. Curved surface = perimeter of base × height.
- III. Whole surface = curved surface + 2 area of an end.

[The Student must bear in mind that all the cylinders and prisms mentioned in the following Examples are right-prisms and right-cylinders.]

EXAMPLES.

Find the volume of a cylinder, when the area of its base and its height are, respectively—

(1) Area of base 67 sq. in., and its height 12 in.

(2) Area of base 3 sq. ft. 30 sq. in., and height 1 ft. 5 in.

(3) Area of base 10 sq. ft. 72 sq. in., and height 12 ft.

Find the volume of a prism, when the area of its base and its height are, respectively—

(4) Area of base 23·5 sq. in., and height 5·4 in.

(5) Area of base 3 sq. ft. 68 sq. in., and height 5 ft. 6 in.

(6) Area of base 2 sq. yds. 2 sq. ft., and height 3 yds.

Find the volume, and also the curved surface, of a cylinder, when the dimensions given are—

(7) Diameter of base 14 in., and height 10 in.
(8) Diameter of base 2 ft. 4 in., and height 5 ft.
(9) Diameter of base 8 ft. 2 in., and height 4 ft. 6 in.
(10) Diameter of base 9 ft. 4 in., and height 12 ft.
(11) Circumference of base 7 ft. 4 in., and height 10 ft.

Find the volume of the following triangular prisms, when the dimensions given are, respectively—

(12) Sides of the base 4, 13, and 15 in., and height 14 in.
(13) Sides of the base 9, 10, and 17 ft., and height 20 ft.
(14) Sides of the base 10, 17, and 21 ft., and height 16 ft.

Find the radius of the base of a cylinder which has the following volume and height:—

(15) Volume 1540 cub. in., and height 10 in.

(16) Volume 19 cub. ft. 1664 cub. in., and height 4 ft. 8 in.

(17) Volume 404 cub. ft. 432 cub. in., and height 10 ft. 6 in.

Find the height of a cylinder, when its volume and the radius of its base are, respectively—

(18) Volume 10 cub. ft. 1200 cub. in., and radius of base 1 ft. 2 in.

(19) Volume 419 cub. ft. 384 cub. in., and radius of base 4 ft. 1 in.

(20) Volume 114 cub. yds. 2 cub. ft., and radius of base 7 ft.

(21) Find the solid contents of a prism whose ends are squares, each side measuring 1 ft. 6 in., and the length of the prism is 14 ft.

(22) Find the volume of a prism whose ends are equilateral triangles, each side measuring 6 ft., and the length of the prism is 20 ft.

(23) Find the volume of a prism whose ends are triangles, whose sides measure, respectively, 12 ft., 17 ft., and 25 ft., and the length of the prism is 24 ft.

(24) Find the volume of an hexagonal prism whose length is 10 ft., the side of the hexagon being 10 in.

(25) Find the volume of a prism 32 ft. long, when its ends are equal trapezoids, the parallel sides being, respectively, 12 ft. and 8 ft., and the perpendicular distance between them is 6 ft.

(26) Find the number of square feet of plastering in the walls and ceiling of a room, in the shape of a cylinder, whose diameter is 21 ft., and the height of the room is 25 ft.

(27*) How many cubic yards must be dug out to make a well 3 ft. in diameter and 30 ft. deep?

(28**) The diameter of a well is 4 ft. 8 in., and its depth 30 ft.; find the cost of excavating it, at 7*s.* 6*d.* per cub. yd.

(29**) What is the expense of painting the walls of a cylindrical room 16 ft. high, and 18 ft. in diameter, at $7\frac{1}{2}d.$ per sq. yd.?

(30*) Find the number of cubic yards of earth dug out of a tunnel 100 yds. long, whose section is a semicircle with a radius of 10 ft.

(31*) The expense of excavating a circular well, of which the depth was 45 ft., and the diameter 3 ft. 9 in., was £8 11*s.* $10\frac{1}{2}d.$; what was the charge per cub. yd.?

(32) A cubic foot of brass is to be drawn into a wire $\frac{1}{11}$ of an inch in diameter; find the length of the wire.

(33) The diameter of a cylindrical vessel is 2 ft. 4 in., and its height is 6 ft. Supposing the diameter were made half as large again, whilst the height remained the same; find what increase there will be in its volume.

(34*) What would it cost to sink a cylindrical well 50 ft. in depth, its diameter being 3 ft. 9 in., at 7*s.* 3*d.* per cub. yd.?

(35*) A cylindrical well is sunk to the depth of 119 ft.;

it measures 4 ft. across at the mouth. How many cubic feet of earth will be taken out?

(36*) The top of a circular table is 7 ft. in diameter and 1 in. thick; how many cubic feet of wood does it contain, and what will it cost to polish the upper surface, at 6*d.* per sq. ft.?

(37) How many cubic inches of mahogany will be required for veneering the top of a table, in the shape of a regular hexagon, each side of which measures 2 ft., the veneer being $\frac{1}{8}$ of an inch thick?

(38) Find the cost of digging a cylindrical tank whose diameter is 14 ft., and whose depth is 20 ft., at $\frac{1}{4}$*d.* per gallon, supposing that 1 cub. ft. contains $6\frac{1}{4}$ gallons. Find also the expense of cementing the sides and bottom, at 8*d.* per sq. ft.

(39) Find the weight of a cylindrical iron shell 1 in. thick and 2 ft. long, whose inner radius is 7 in., when iron weighs 7·2 times as much as water, and 1 cub. ft. of water weighs 1000 oz. avoirdupois.

(40) The water in a cylindrical well, whose diameter is 7 ft., is found to stand to the height of 10 ft.; find how many gallons of water there are in the well, reckoning $6\frac{1}{4}$ gallons to the cubic foot.

(41) An open cylindrical vessel, whose depth is 7 ft., is constructed to hold 198 cub. ft. of water; find how many cubic feet of lead will be required for lining its curved surface and bottom, when the lead is $\frac{1}{8}$ of an inch thick.

(42) An open cylindrical vessel, which measures 3 ft. 6 in. across at the top, requires 119 sq. ft. 90 sq. in. of lead for lining its curved surface and bottom; find how many cubic feet of water the vessel will contain.

(43) A cast-iron pipe, 2 in. thick, whose internal radius is 7 in., and whose length is 5 ft., is to be melted and cast into another pipe of the same thickness, but whose internal radius is only 4 in.; find the length of the new pipe.

(44) The whole internal surface of an open cylindrical

vessel, whose radius is 5 ft. 10 in., is 565 sq. ft. 40 sq. in.; find how many gallons of water the vessel will hold, supposing that a cubic foot contains $6\frac{1}{4}$ gallons of water.

(45) A marble column measures 7 ft. 4 in. in circumference, and its height is 15 ft.; find the expense of polishing the curved surface, at 3*s.* 4*d.* per sq. ft., and also the additional expense if the whole surface were polished.

(46) A cubic foot contains $6\frac{1}{4}$ gallons. Having pumped 310 gallons of water out of a well, whose diameter is 7 ft., find how many inches the water in the well will have sunk in consequence.

(47*) What will be the cost of making a cylindrical cask, closed at both ends, 3 ft. high, and 3 ft. in diameter, the price of the wood being 6*d.* per sq. ft., and the charge for labour being made at the rate of 3*d.* per gallon on the contents of the barrel? (Take a gallon=270 cub. in.)

(48**) The Wall of China is 1500 miles long, 20 ft. high, 15 ft. wide at the top, and 25 ft. wide at the bottom; how many cubic yards of material does it contain?

(49*) How many pieces of money, $\frac{3}{4}$ of an inch in diameter, and $\frac{1}{8}$ of an inch thick, must be melted down to form a cube whose edge is 3 in. long?

(50**) What length of wire ·08 of an inch thick can be formed out of a cubic inch of metal?

(51*) A square iron rod, an inch thick, and a yard long, weighs $10\frac{1}{2}$ lbs. How much would a round iron rod, of the same length and thickness, weigh?

(52) How many cubic feet of metal would be required for making a wire $\frac{1}{11}$ of an inch thick which would stretch from London to Edinburgh, a distance of 405 miles?

(53*) How many revolutions of a roller, 3 ft. in length, and 18 in. in diameter, would it take to go over a grass-plot half an acre in extent?

XXI. THE RIGHT-PYRAMID AND THE RIGHT-CONE.

Definitions.—I. A pyramid is a solid having any plane figure for its base, and triangles for its sides, all of which terminate in one common point, called the *vertex.* The pyramid takes particular names, according to the shape of the base. If the base is a triangle, then it is called a triangular pyramid. If the base has six sides, then it is called an hexagonal pyramid; and so on. A *right-pyramid* (fig. 1) is a pyramid in which the straight line C G, drawn from the centre C of the base to the vertex G, is *perpendicular* to the base. This line C G is the *axis* of the pyramid.

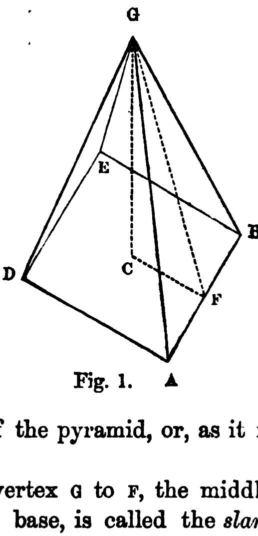


Fig. 1.

C G is the *perpendicular height* of the pyramid, or, as it is simply called, the *height.*

G F, the line drawn from the vertex G to F, the middle point of one of the sides of the base, is called the *slant height* of the pyramid.

G A, G B, G E, &c. are called the *edges* of the pyramid, which meet in the vertex G.

The Student must particularly bear in mind that G F is the *slant height,* and not G A, G B, &c.

The *oblique pyramid* is one in which the axis C G is *not perpendicular* to the base of the pyramid. The *height* of an *oblique pyramid* is not the axis G C, but is some perpendicular drawn from the vertex G to the base, which will require to be found before we can find the volume of the pyramid.

The *whole surface* of a pyramid, whether right or oblique

consists of the area of the base and also the areas of the triangular side faces.

II. A cone is a solid which has a circular base, and whose curved surface terminates in a point, called the *vertex*.

The line CG, joining the vertex G with C, the centre of the circular base, is called the *axis* of the cone.

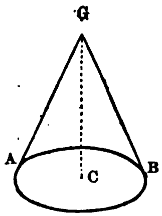


Fig. 2.

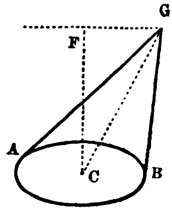


Fig. 3.

A right-cone (fig. 2) is one in which the axis GC is *perpendicular* to the base.

GC is the *perpendicular height*, which is necessary to be known before finding the volume.

GA or GB is the *slant height*, which is necessary to be known before finding the curved surface.

An oblique cone (fig. 3) is one in which the axis GC is *not perpendicular* to the base; in which case CF, and not CG, is the perpendicular height.

The *whole surface* of a cone will consist of the area of the circular end and the area of the curved or round surface.

N.B.—All questions in this chapter refer to the right-cone and the right-pyramid.

RULES.—(1) *To find the volume of a right-cone or right pyramid.*

Multiply the area of the base by the perpendicular height, and divide the product by 3.

(2) *To find the curved surface of a right-cone or side faces of a right-pyramid.*

Multiply the perimeter of the base by the slant height, and divide the product by 2.

(3) *To find the whole surface of a right-cone or right-pyramid.*

To the area of the curved surface or side faces (found by Rule 2), add the area of the base.

Note 1.—To find the height of a right-cone or right-pyramid, when its volume and the area of its base are given.—Divide three times the volume by the area of the base.

Note 2.—To find the area of the base of a right-cone or right-pyramid, when its volume and height are given.—Divide three times the volume by the height.

Note 3.—The volume of a right-cone is one-third of the volume of a right-cylinder having the same base and the same height; and the volume of a right-pyramid is one-third of the volume of a right-prism having the same base and the same height.

Note 4.—The volume of an oblique cone DAB is equal to the volume of a right-cone GAB, having the *same area* for the base, and lying between the *same parallel planes*. Hence, the volume of an oblique cone (fig. 4) is found by multiplying the area of the circular base by the perpendicular height CG (and not by the axis CD). With regard to the *curved surface* of an *oblique cone*, it can be shown that the greater the *obliquity* is, the greater is the *curved surface*. No simple expression can be given for finding the *curved surface* of an *oblique cone*. The same remarks will apply to the case of an oblique pyramid.

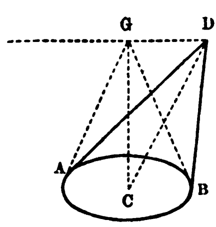


Fig. 4.

Example 1.—Find the volume of a cone, when the circumference of its base is 21 ft., and its height is 24 ft.

Area of base, by Rule 2, Prob. XII. $=21^2 \times \frac{7}{88} = \frac{3087}{88}$ sq. ft.

Then, volume of cone $= \frac{3087}{88} \times 24 \times \frac{1}{3} = \frac{3087}{11}$ cub. ft. $=$ $280\frac{7}{11}$ cub. ft.

Example 2.—Find the curved surface of a cone whose diameter is 10 ft. 6 in., and slant height is 18 ft.

The circumference of base, by Rule 1, Prob. XI. $=10$ ft. 6 in. $\times \frac{22}{7} = 33$ ft.

Then, curved surface $=33 \times 18 \times \frac{1}{2} = 297$ sq. ft.

Example 3.—The volume of a triangular pyramid is 792 cub. ft., and its height is 36 ft.; find the area of the base.

Now, $792 \times 3 = 2376$ cub. ft.

Then, by Note 2, Area of base $=2376 \div 36 = 66$ sq. ft.

Example 4.—The base of a pyramid is a square, each side of which is 24 ft.; the length of the straight line drawn from the vertex to the middle point of any side of the base is 37 ft. Find the volume.

Referring to fig. 1, we have the right-angled triangle GCF, of which the hypothenuse GF is 37 ft., and the base CF (half the length of a side of the base) is 12 ft.; and it is necessary to find, in the first place, the perpendicular CG, which is $= \sqrt{37^2 - 12^2} = 35$ ft.

Then, volume of pyramid $=$ area of base $\times 35 \times \frac{1}{3}$

$$= 24 \times 24 \times 35 \times \frac{1}{3} = 6720 \text{ cub. ft.}$$

Formulæ:

I. Volume $=$ area of base $\times$ perpendicular height $\times \frac{1}{3}$.

II. Curved surface $=$ perimeter of base $\times$ slant height $\times \frac{1}{2}$.

III. Whole surface $=$ curved surface $+$ area of base.

IV. $\text{Height} = \dfrac{3 \text{ volume}}{\text{area of base}}$

V. $\text{Area of base} = \dfrac{3 \text{ volume}}{\text{height}}$

[Observe that all the cones and pyramids mentioned in the following Examples are right-cones and right-pyramids; and that when *height* or *altitude* is mentioned, it refers to the *perpendicular height*, unless otherwise stated. Also bear in mind that it is necessary to know the *perpendicular height* before finding the *volume* of a cone or pyramid; but the *slant height* in finding the *curved surface.*]

EXAMPLES.

Find the volume of a cone, when the area of its base and its height are, respectively—

(1) Area of base 90 sq. in., and height 3 ft. 6 in.
(2) Area of base 2 sq. ft. 12 sq. in., and height 12 ft.
(3) Area of base 11 sq. ft. 24 sq. in., and height 18 ft.

Find the volume of a cone, when the dimensions are, respectively—

(4) Diameter of base 14 in., and height 3 ft. 6 in.
(5) Diameter of base 7 ft., and height 10 ft.
(6) Diameter of base 5·6 ft., and height 14 ft.
(7) Circumference of base 14 ft. 8 in., and height 9 ft.
(8) Circumference of base 29 ft. 4 in., and height 21 ft.

Find the volume, and also the curved surface, of the following cones, when the dimensions are, respectively—

(9) Radius of base 3 ft. 6 in., and slant height 5 ft. 10 in.
(10) Radius of base 2 ft. 4 in., and slant height 4 ft. 5 in.
(11) Radius of base 12 ft., and slant height 37 ft.
(12) Circumference of base 44 ft., and slant height 25 ft.

Find the volume of a pyramid, when the area of its base and its height are, respectively—

(13) Area of base 64 sq. in., and height 1 ft. 9 in.
(14) Area of base 2 sq. ft. 36 sq. in., and height 6 ft.

Find the volume of a pyramid with the following dimensions :—

(15) When its base is a square, each side measuring 3 ft. 4 in.; and height 9 ft.

(16) When its base is an equilateral triangle, each side measuring 4 ft.; and height 15 ft.

(17) When its base is a regular hexagon, each side measuring 6 ft.; and height 30 ft.

Find the *whole* surface of the following pyramids :—

(18) When each side of its square base is 8 ft., and the slant height is 20 ft.

(19) When each side of its triangular base is 6 ft., and the slant height is 18 ft.

(20) When each side of its square base is 26 ft., and the perpendicular height is 84 ft.

Find the diameter of the base of a cone, when its volume and height are, respectively—

(21) Volume 51 cub. ft. 576 cub. in., and height 9 ft.

(22) Volume 492·8 cub. ft., and height 15 ft.

Find the height of a pyramid, when its volume and its base are, respectively—

(23) Volume 26 cub. ft. 936 cub. in., and each side of its square base is 3 ft. 6 in.

(24) Volume 20 cub. ft., and the sides of its triangular base are, respectively, 5 ft., 4 ft., and 3 ft.

(25) The area of the curved surface of a cone is 99 sq. ft., and the radius of its base is 2 ft. $7\frac{1}{2}$ in.; find its slant height.

(26) The slant height of a cone, whose curved surface measures 2310 sq. ft., is 35 ft.; find its height.

(27) The length of the line drawn from the vertex of a pyramid to the corner of its square base, each side of which measures 40 ft., is 101 ft.; find its volume.

(28) The volume of a pyramid, whose base is a square, each side of which measures 5 ft., is 200 cub. ft.; find the whole surface.

(29) How much ground will the base of a pyramid cover which contains 102 cub. yds. 21 cub. ft. of earth, and whose height is 45 ft.?

(30) A conical mound of earth measures 264 yds. all round the base, and the length of its slope, from the foot to its summit, is 70 yds.; find the number of cubic yards in the mound.

(31) Find the height of a conical vessel, whose diameter at the base is 2 ft. 4 in., in order that its volume may be the *same* as that of a cubical vessel which measures 3 ft. every way.

(32) How many cubic feet of lead will be required for covering a conical spire which measures 35 ft. round at the base, and whose slant height is 30 ft.; supposing that the lead is $\frac{1}{12}$ of an inch thick.

(33) A cubic inch of gold weighs 11 oz. avoir.; find what weight will be used in making a solid gold ornament, in the shape of a square pyramid, each side of its base being 3 in. long, and its height 6 in.

(34) Find the volume of a pyramid with an equilateral triangular base, each side of which measures 8 ft., and whose height is 33 ft.; supposing that there is a hollow part in the middle, in the shape of a cone, whose diameter is 7 ft. and height 21 ft.

(35) Find the expense of covering a conical spire which measures 40 ft. round the base, and whose slant height is 30 ft., with lead which is $\frac{1}{8}$ of an inch thick; supposing that 1 cub. in. of lead weighs $6\frac{1}{2}$ oz. avoir., and that lead is worth 4*d.* per lb.

(36**) Find the volume of a right cone, the height of which is 15 ft., and the circumference of the base is 14 ft.

(37*) How much canvas is necessary for a conical te

the altitude of which is 8 ft., and the diameter of the base 7 ft.?

(38*) The circumference of the base of a cone is 12 ft. 6 in., and its perpendicular height is 8 ft. 3 in.; find its volume.

(39) A conical tent, whose slant height is 12 ft., requires 132 sq. ft. of canvas to cover it; find how much ground the tent covers.

(40**) A right-angled triangle, of which the sides are 3, 4, and 5 in. in length, is made to turn round on the side, the length of which is 4 in.; find the curved surface and the volume of the cone.

(41**) What is the whole superficies of a square pyramid, each side of the base of which is 12 ft., and its slant height is 25 ft.?

(42**) What will be the cost of painting a conical spire, whose slant height is 118 ft., and whose circumference at the base is 64 ft., at 8*d.* per square yard?

(43*) A cone, 3 ft. high and 2 ft. in diameter at the bottom, is placed on the ground, and sand is poured over it until it makes a conical heap 5 ft. high, and 30 ft. in circumference; how many cubic feet of sand are there?

(44**) How many gallons are contained in a vessel, which is in the shape of a right cone, the radius of the base being 6 ft., and the length of the slant height 10 ft.? (1 gallon=$277\frac{1}{4}$ cub. in.)

(45**) A pyramidal roof 16 ft. high, standing on a square base, each side of which is 24 ft., is covered with sheet-lead $\frac{1}{16}$ of an inch thick; what is the weight of the lead, supposing a cubic inch to weigh 7 oz. avoirdupois?

(46*) What length of canvas, $\frac{3}{4}$ yd. wide, is required to make a conical tent 12 ft. in diameter, and 8 ft. high?

(47*) What length of canvas, $\frac{2}{3}$ yd. wide, is required to cover a conical tent 16 yds. in diameter, and 10 ft. high?

(48*) A conical wine-glass is 2 in. wide at the top, and deep; how many cubic inches of wine will it hold?

(49**) How many yards of canvas, $\frac{3}{4}$yd. wide, will be required to make a conical tent 8 ft. high, and 10 ft. in diameter? How many cubic feet of air will the same contain?

(50*) A cylindrical stick, $\frac{1}{3}$ in. thick, is sharpened at one end into the shape of a cone, the slant side of which is $\frac{5}{18}$ of an in. long; how much wood is cut away in doing this?

(51) The radius of a cylinder is 7 ft., and its height is 12 ft., find the perpendicular height of a cone that shall have the *same* base and area of curved surface as the cylinder.

(52*) A square tower, 21 ft. on each side, is to have either a flat roof covered with sheet-lead which costs 6*d.* per sq. ft., or a pyramidal roof, whose vertical height is 10 ft., covered with slates which cost 18*s.* 9*d.* per hundred, and each of which has an exposed surface of 10 in. by 9 in.; find the cost in each case.

XXII. THE FRUSTUM OF A RIGHT-CONE OR RIGHT-PYRAMID.

Definitions.—The frustum of a cone or pyramid is that part of the cone or pyramid which remains when its top part has been cut off by a plane parallel to the base. The top part cut off, in the one case, is a cone, and in the other a pyramid.

The base AB and its opposite face DE are called the *ends* of the frustum.

GF, the perpendicular drawn from one end to the other, is called the *perpendicular height*; or, more generally, simply the *height* of the frustum.

AD is, in the case of the cone (fig. 1), the *slant height*, and is the hypothenuse of a right-angled triangle DLA, of

which the perpendicular is DL, the height of the frustum; and the base is AL, the difference between the radii of the two ends.

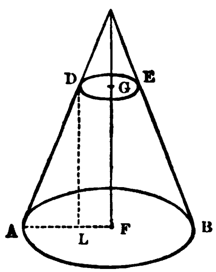


Fig. 1.

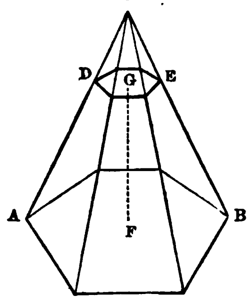


Fig. 2.

The *slant height* of the frustum of a pyramid (fig. 2) is measured from the *middle* of a side of one end to the *middle* of the corresponding side of the other end.

The *whole surface* of the frustum of a cone (fig. 1) is the areas of the two circular ends added to the area of the curved surface.

The *whole surface* of the frustum of a pyramid (fig. 2) is the areas of the two ends added to the area of the side faces.

[N.B.—The attention of the Student in this chapter is entirely confined to the consideration of the frustum of a *right*-cone and of a *right*-pyramid.]

RULES.—(1) *To find the volume of the frustum of a cone or pyramid.*

To the areas of the two ends of the frustum add the square root of their product; multiply the sum by the height of the frustum; and one-third of the product will be the volume.

(2) *To find the volume of the frustum of a cone, when the diameter or radii of the two ends are given.*

(*a*) Add the squares of the radii of the ends to the product of the radii; multiply the sum by the height, and this product by $\frac{22}{7}$; one-third of the result is the volume.

(*b*) Or, add the squares of the diameters of the ends to the product of the diameters; multiply the sum by the height, and this product by $\frac{11}{14}$; one-third of the result is the volume.

(3) *To find the curved surface of the frustum of a cone, or the area of the side faces of the frustum of a pyramid.*

Add together the circumferences or perimeters of the two ends; multiply the sum by the slant height of the frustum, and half the product is the area.

(4) *To find the whole surface of the frustum of a cone or pyramid.*

To the area of the curved surface or side faces, add the areas of the two ends.

Example 1.—Find the volume of the frustum of a cone, when the diameters of its ends are, respectively, 16 ft. and 9 ft., and the height of the frustum is 24 ft.

By Rule 1: Area of one end$=16^2\times\frac{11}{14}=256\times\frac{11}{14}$; the area of the other end $=9^2\times\frac{11}{14}=81\times\frac{11}{14}$; the square root of the product of these areas $=\sqrt{256\times\frac{11}{14}\times 81\times\frac{11}{14}}$ $=\frac{11}{14}\sqrt{256\times 81}=\frac{11}{14}\times 16\times 9=\frac{11}{14}\times 144$. Add these results, and we have $\frac{11}{14}$ multiplied by the sum of 256, 81, and 144; that is, $\frac{11}{14}\times 481$.

Then, volume$=\frac{1}{3}\times 24\times 481\times\frac{11}{14}=\frac{21164}{7}=3023\frac{3}{7}$ cub. ft.

Example 2.—Find the volume of the frustum of a square pyramid, each side of one of its ends being 8 ft., and each side of the other end 4 ft., and the perpendicular height is 10 ft.

The area of one end$=8^2=64$; the area of the other end $=4^2=16$; and the square root of the product of these areas$=\sqrt{64\times16}=8\times4=32$.

Adding together 64, 16, and 32, we have 112.

Then, volume$=112\times10\times\frac{1}{3}=373\frac{1}{3}$ cub. ft.

Example 3.—Find the curved surface of the frustum of a cone whose slant height is 16·5 ft., and the circumference of its two ends are, respectively, 14·2 ft. and 11·8 ft.

The sum of the circumferences of two ends$=14\cdot2+11\cdot8$ $=26$ ft.

Then, curved surface$=26\times16\cdot5\times\frac{1}{2}=214\cdot5$ sq. ft.

Formulæ:

I. Volume$=\{$area of one end$+$area of other end$+\sqrt{}($area of one end$\times$area of other end$)\}\times\frac{\text{height}}{3}$

II. Volume$=\{$radius of one end$^2+$radius of other end$^2+$product of two radii$\}\times\frac{22}{7}\times\frac{\text{height}}{3}$

III. Volume $=\{$diameter of one end^2 $+$diameter of other end$^2+$product of two diameters$\}\times\frac{11}{14}\times\frac{\text{height}}{3}$

IV. Curved surface $=\{$circumference of one end $+$ circumference of other end$\}\times\frac{\text{slant height}}{2}$

V. Whole surface$=$curved surface$+$areas of two ends.

[Observe that the following questions refer only to the frustum of a right-cone or to that of a right-pyramid,

and that it is necessary to know the *perpendicular* height before finding the volume, and the *slant* height before finding the curved surface. When the word height occurs *alone*, it refers to the *perpendicular* height.]

EXAMPLES.

Find the volume of the frustum of a cone, when the dimensions given are—

(1) Radii of circular ends 5 ft. and 4 ft., and height 6 ft.

(2) Radii of circular ends 8 ft. and 6 ft., and height 15 ft.

(3) Radii of circular ends 10 ft. 6 in. and 7 ft., and height 12 ft.

(4) Radii of circular ends 3 ft. 10 in. and 3 ft., and slant height 2 ft. 2 in.

(5) Radii of circular ends 6 ft. 5 in. and 3 ft. 6 in., and slant height 7 ft. 7 in.

Find the curved surface of the frustum of a cone, when its dimensions are, respectively—

(6) Radii of circular ends 6 ft. and 5 ft., and slant height 14 ft.

(7) Radii of circular ends 14 ft. and 7 ft., and slant height 30 ft.

(8) Radii of circular ends 14 ft. and 6 ft., and height 15 ft.

(9) Radii of circular ends 14 ft. and 7 ft., and height 24 ft.

(10) Find the volume of the frustum of a square pyramid whose height is 24 ft., and the sides of its square ends are, respectively, 9 ft. and 4 ft.

(11) The length of each side of the base of the frustum of a square pyramid is 16 ft., and of each side of the top is 9 ft.; the height of the frustum is 30 ft. Find its volume.

(12) The slant height of the frustum of a square

pyramid is 20 ft., the length of each side of its square base is 40 ft., and of each side of its top is 16 ft.; find its volume.

(13) The slant height of the spire on a church tower, in the shape of the frustum of an hexagonal pyramid, is 20 ft.; the length of each side of its base is 5 ft., and of its top 2 ft. How many square feet of lead will be required for covering its side faces and top?

(14) Find the expense of polishing the curved surface of a column of marble, in the shape of the frustum of a cone, whose slant height is 12 ft., and the radii of the circular ends are 3 ft. 6 in. and 2 ft. 4 in. respectively, at 2*s*. 6*d*. per sq. ft.

(15) Find the cost of the mast of a ship, in the shape of the frustum of a cone, at 2*s*. 6*d*. per cub. ft., when its perpendicular height is 51 ft., and the circumference at one end is 5 ft. 6 in., and at the other end 1 ft. 10 in.

(16) A bucket is 1 ft. 4 in. deep; its diameter at the top is 1 ft. 6 in., and at the bottom 1 ft. Find how many gallons of water it will hold, when there are $6\frac{1}{4}$ gallons to a cub. ft.

(17) How many cubic inches of lead will be required for covering the curved surface and the bottom of a vessel, in the shape of the frustum of a cone, with lead $\frac{1}{16}$ of an in. thick; the diameter of the top of the vessel is 7 ft., and of the bottom 3 ft. 6 in., and the perpendicular depth is 2 ft. 4 in.?

(18) Supposing that an ordinary glass tumbler, in the shape of the frustum of a cone, is $3\frac{1}{2}$ in. wide at the top, and $2\frac{1}{2}$ in. wide at the bottom, and that its depth is $4\frac{1}{2}$ in.; find how many cubic inches of water it will hold.

(19) A marble column, in the shape of the frustum of a square pyramid, is 18 ft. high; the length of each side of its base is 4 ft., and of its top $2\frac{1}{4}$ ft. Find its volume.

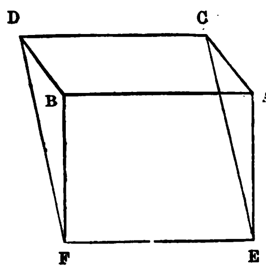
(20) An iron vessel without lid, in the shape of the frustum of a cone, has the following *internal* dimensions:—The diameter at the top is 1 ft. 9 in., at the bottom 7 in., and the depth 1 ft. 8 in.; whilst its corresponding *external*

dimensions are 2 ft., 9 in., and 1 ft. 10 in. Find how many cubic inches of iron were used in its construction.

(21*) Find the content of a cask, in the form of a conical frustum, the radii of its ends being 2 ft. and 3 ft., and the height 5 ft.

XXIII. THE WEDGE.

Definitions.—A wedge is a solid of which the *base* or *thick end* ABCD is a rectangle; the two *ends* AEC, BFD are triangles; and the two opposite faces AEFB, CEFD meet in the *edge* or *thin end* EF, and are either rectangles or trapezoids.



In the *common wedge* we have EF=AB=CD, and the solid becomes a triangular prism, of which its ends AEC, BDF are equal and parallel isosceles triangles, and its base and two side faces are rectangles.

The *volume* of a *common wedge* is the *same* as that of a *triangular prism*, and may therefore be found by multiplying the area of its end BFD by EF, the length of the wedge.

But when EF is not equal to AB or CD, then the two side faces EABF, ECDF are trapezoids; and the volume of such a wedge is found by the rule given below.

The *perpendicular height*—or, as it is often called simply, the *height*—is the *perpendicular* drawn from any point in the *edge* EF to the *base* ABCD.

[In some cases the base is a trapezoid; but it is desirable that the Student should confine his attention at present to the consideration of the simpler kinds of the wedge.]

RULES.—(1) *To find the volume of a wedge.*

To twice the length of the base or thick end, add the length of the thin end or edge; multiply the sum by the breadth of the base or thick end, and the product by the height of the wedge; one-sixth of the result is the volume of the wedge.

(2) *To find the whole surface of a wedge.*

Find the area of each of its sides or parts, and add them together.

Note.—To find the volume of a wedge, when the area of a section of it, perpendicular to the edge, and also the length of its base and edge, are given.—Add together the edge and twice the length of the base, and divide the sum by 3, and the quotient is the *mean length* of the wedge.

Multiply the area of the section by the mean length, and the product is the volume of the wedge.

Example 1.—Find the volume of a wedge, when the length and breadth of its base are 26 in. and 18 in., the length of the thin end or edge is 15 in., and the height is 28 in.

Now, $\overline{26 \times 2 + 15} = 52 + 15 = 67$ in.

Then, volume of wedge $= 67 \times 18 \times 28 \times \frac{1}{6} = 5628$ cubic in. $= 3$ cub. ft. 444 cub. in.

Example 2.—The edge of a wedge is 110 in.; the length of the base is 70 in., and its breadth is 30 in.; the height of the wedge is 24·8 in. Find its volume.

Now, $\overline{2 \times 70 + 110} = 140 + 110 = 250$ in.

Then, volume of wedge $= 250 \times 30 \times 24{\cdot}8 \times \frac{1}{6} = 31000$ cub. in. $= 17$ cub. ft. 1624 cub. in.

Formulæ $\begin{cases} \text{I. Volume} = (\overline{2\text{AB} + \text{EF}} \times \text{AC} \times \text{height} \times \frac{1}{6}). \\ \text{II. Whole surface} = \text{sum of the areas of all its sides or parts.} \end{cases}$

[N.B.—When the word *height* occurs alone, it refers to the *perpendicular* height—that is, to the straight line drawn from the edge perpendicular to the base.]

EXAMPLES.

(1) Find the volume of a wedge, of which the base is a rectangle, whose length is 12 in. and breadth 6 in.; the edge is 12 in., and the height of the wedge is 10 in.

(2) The edge of a wedge is 1 ft. 8 in.; the base of the wedge is a square, each side of which measures 1 ft. 8 in.; the height of the wedge is 3 ft. Find its volume.

(3) Find the volume of a wedge, whose base is 3 ft. 6 in. long and 1 ft. 8 in. broad; the length of the edge is 2 ft., and the height of the wedge is 2 ft. 6 in.

(4) The length of the rectangular base of a wedge is 20 in., and its breadth is 10 in.; the edge is 11 in., and the height of the wedge is 2 ft. Find its volume.

(5) The edge of a wedge is 1 ft. 3 in.; the length of the base is 2 ft.; the area of a section of the wedge made by a plane perpendicular to the edge is 1 sq. ft. 36 sq. in. Find the volume of the wedge.

(6) The volume of a wedge is 1020 cub. in.; its edge is 15 in.; its base measures 18 in. in length and 12 in. in breadth. Find its height.

(7) Find the volume of a wedge, when the following dimensions are given:—

AB = 4 ft. 6 in.; EF = 3 ft.; AC = 1 ft. 4 in.; and AE = 1 ft. 5 in.

(8) Find the volume of a wedge, when the following dimensions are given:—

AB = 6·5 ft.; EF = 8 ft.; AC = 4·25 ft.; and the line drawn from E to the middle of AC = 12 ft.

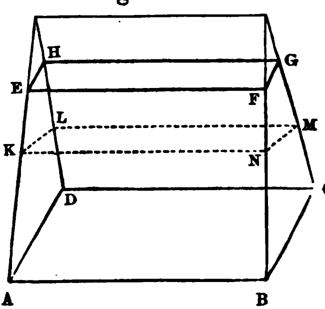
(9**) The length of the edge of a wedge is $5\frac{1}{2}$ in.; the length of the base is 3 in., and its breadth is 2 in.; the height of the wedge is 4 in. Find its volume.

(10*) Find the solid contents of a wedge whose edge and length of the base are each 18 in.; the breadth of the base is 8 in., and the height 30 in.

(11) Find the whole surface of a wedge, when its base is 5 ft. 6 in. long and 2 ft. broad; the edge is 5 ft. 6 in., and the height of the wedge is 2 ft. 11 in.

XXIV. THE PRISMOID, OR FRUSTUM OF A WEDGE.

Definitions.—A prismoid, or frustum of a wedge, is the solid that remains after a smaller wedge has been cut off by a plane parallel to the base.



Thus ABGE is a prismoid, having its ends BD and HF rectangles, its side faces EB and HC either rectangles or trapezoids, and its side faces ED and FC always trapezoids.

The side face ABCD is also called the *base*; then its opposite face HEFG is called the *top*. Both these side faces are sometimes called the *ends* of a prismoid.

The *perpendicular height*—or, as it is generally called simply, the *height*—is the *perpendicular* drawn from *one end* HF to the *other end* BD.

If LKNM is the *middle section* of the prismoid—that is, a section parallel to and equally distant from each end—then its length KN or $ML = \frac{AB+EF}{2}$; and its breadth LK or MN $= \frac{AD+EH}{2}$.

RULES.—(1) *To find the volume of a prismoid.*

Add together the areas of the two ends and four times the area of a section parallel to, and equally distant from, both ends; multiply the sum by the perpendicular height, and one-sixth of the product is the volume.

(2) *To find the whole surface of a prismoid.*

Add together the areas of the two ends and of the four side faces.

Note.—The *length* of the *middle section* is equal to half the sum of the lengths of the two ends; and its *breadth* is equal to half the sum of the breadths of the two ends; and the *area* of the *middle section* is equal to one-fourth the area of a rectangle having for its length the sum of the lengths of the two ends, and for its breadth the sum of the breadths of the two ends.

Thus, four times the area of the middle section, mentioned in the rule, is, in fact, the area of a rectangle having for its length the sum of the lengths of the two ends, and for its breadth the sum of the breadths of the two ends.

Example 1.—The length and breadth of the lower end of a prismoid are 12 ft. and 10 ft., and those of the upper end are 15 ft. and 7 ft.; the height is 9 ft. Find the volume of the prismoid.

The length of the middle section $=\frac{12+15}{2}$; and its breadth$=\frac{10+7}{2}$; so that four times its area is$=27\times17=$ 459 sq. ft.

The area of the lower end$=12\times10=120$ sq. ft.; and area of upper end$=15\times7=105$ sq. ft.

Hence, volume of prismoid$=(120+105+459)\times9\times\frac{1}{6}=$ 1026 cub. ft.

Example 2.—The length and breadth of a reservoir, in the shape of a prismoid, are 140 ft. and 80 ft.; the length and breadth at the bottom are 100 ft. and 60 ft.; the depth is 12 ft. Find how many cubic feet of earth were dug out.

The area of the top $=140\times 80=11200$ sq. ft.

Area of the bottom $=100\times 60=6000$ sq. ft.

Also, length of middle section $=\dfrac{140+100}{2}=120$ ft.; and its breadth $=\dfrac{80+60}{2}=70$ ft.; then four times area of this section $=4\times 120\times 70=33600$ sq. ft.

Therefore volume $=(33600+11200+6000)\times 12\times \frac{1}{6}=$ 101600 cub. ft.

Formulæ:
- I. Volume = (area of top + area of bottom + four times area of middle section) × height × $\frac{1}{6}$.
- II. Whole surface = area of top + area of bottom + area of four side faces.

[Observe that when the word *height* occurs alone, it refers to the *perpendicular* height.]

EXAMPLES.

(1) How many cubic feet are contained in a prismoid whose height is 2 ft. 6 in. and whose top and bottom are rectangles, the lower rectangle measuring 40 in. by 28 in., and the upper one 20 in. by 14 in.?

(2) Find the volume of a prismoid whose bottom is a rectangle measuring 56 ft. by 40 ft.; the top is a square measuring 48 ft. every way; the height of the prismoid is 42 ft.

(3) Find the volume of a prismoid whose base is a rectangle measuring 24 ft. by 16 ft., and its top is also a rectangle measuring 20 ft. by 12 ft.; the height is 18 ft.

(4) Find the solid contents of a prismoid whose height

is 12 ft., and whose ends are rectangles, the dimensions of the lower one being 200 ft. by 160 ft., and the dimensions of the upper one 160 ft. by 128 ft.

(5) A railway coal-waggon is in the shape of a prismoid; its top is a rectangle 6 ft. by 4 ft., and its bottom is also a rectangle 4 ft. by 2 ft. 8 in.; its depth is 4 ft. Find its volume.

(6) How many cubic feet of water will a reservoir hold, which is of the uniform depth of 10 ft., whose surface is a rectangle measuring 80 ft. by 50 ft., and whose bottom is also a rectangle which measures 72 ft. by 40 ft.?

(7) A railway cutting is 50 yards long, and its width at the bottom is *uniformly* 30 ft. The ends of the cutting are trapezoids. The width at the top of one end of the cutting is 80 ft., and its depth 30 ft.; whilst the width at the top of the other end is 60 ft., and its depth 20 ft. Find how many cubic feet of earth have been removed.

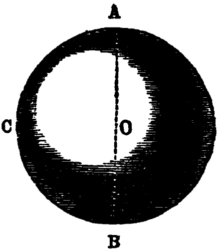
(8) The top and the bottom of a reservoir, in the shape of a prismoid, are rectangles; the dimensions of the top being 200 ft. by 150 ft., and of the bottom 160 ft. by 130 ft.; its uniform depth is 12 ft. Find the cost of excavation, at 1*s.* 6*d.* per cub. yd.

(9) The ends of a railway cutting are trapezoids; the parallel sides of one end 50 ft. and 30 ft., and the distance between them 30 ft.; and of the other end the parallel sides are 40 ft. and 20 ft., and the distance between them is 20 ft.; the length of the cutting is 200 yds. Find the number of cubic feet of earth removed.

(10**) The length of a railway cutting is 6 ch.; the top width and depth at one end 120 ft. and 32 ft. respectively; and at the other 108 ft. and 28 ft.; and the bottom width is 24 ft. throughout. Find the cubic contents.

XXV. THE SPHERE.

Definitions.—A sphere or globe is a solid bounded by a curved surface, every point of which is equally distant from a certain point within the sphere, called the centre.



The *radius* of a sphere is the straight line drawn from the centre to the surface.

The *diameter* of a sphere is the straight line drawn through the centre, and terminated at both ends by the surface.

RULES.—(1) *To find the volume of a sphere.*

Multiply the cube of the diameter by $\frac{22}{7}$, and divide the product by 6; that is,

Multiply the cube of the diameter by $\frac{11}{21}$.

(2) *To find the curved surface of a sphere.*

Multiply the square of the diameter by $\frac{22}{7}$.

Note 1.—To find the diameter of a sphere, when its volume is given.—Divide the volume by $\frac{11}{21}$, and the cube root of the quotient is the diameter.

Note 2.—To find the diameter of a sphere, when its curved surface is given.—Divide the curved surface by $\frac{22}{7}$; and the square root of the quotient is the diameter.

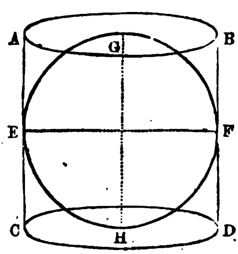
Note 3.—(*a*) The volume of a spherical shell may be found by finding the volume of the *whole* sphere considered as a solid, and then also of the *hollow part* considered as a solid; the difference between these two will be the volume of the shell

(*b*) Or, the following rule may be employed:—Subtract the cube of the inner diameter from the cube of the outer diameter; multiply the remainder by $\frac{11}{21}$, and the product is the volume of the shell.

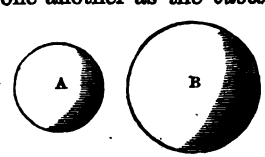
N.B.—The outer diameter = inner diameter + 2 thickness of shell;
and inner diameter = outer diameter − 2 thickness of shell.

Note 4.—The rules above give the answers with *sufficient* accuracy; but should still greater accuracy be desirable, then take 3·1416 as the multiplier, instead of $\frac{22}{7}$.

Note 5.—Of all solids, the sphere is that which contains the *greatest* volume under the *least* surface; that is, if we have a cube, a cylinder, a cone, a sphere, &c., *all* having the *same* surface, then the *sphere* will have the *greatest volume* of all these figures.



Note 6.—If EFGH is a sphere whose diameter is GH or EF, and if ABCD is its circumscribing cylinder—that is, is a cylinder whose height and the diameters of its circular top and bottom are *all* equal to the diameter of the sphere—
Then it is found that the volume of the sphere is $\frac{2}{3}$ of the volume of the cylinder.



Note 7.—(*a*) All spheres are to one another as the *cubes* of their diameters; so that, if the diameter of a sphere is twice that of another, then its volume is 2^3 or 8 times that of the other.

Hence, if it is required to find the diameter of a sphere B-

whose volume is twice that of the sphere A, whose diameter is known, we have

Volume of sphere B $=2$ volume of sphere A.

$$(\text{Diameter of B})^3 \times \tfrac{11}{21} = 2 \times (\text{diameter of A})^3 \times \tfrac{11}{21}.$$

Striking out the factor $\frac{11}{21}$, which is common to both sides, we have

$$(\text{diameter of B})^3 = 2 \times (\text{diameter of A})^3.$$

And since the cube roots of these quantities are also equal, we have

$$\text{diameter of B} = \text{diameter of A} \times \sqrt[3]{2}.$$

Therefore, if the diameter of a sphere A is multiplied by $\sqrt[3]{2}$, we shall have the diameter of another sphere B, whose volume is twice that of the given sphere.

If the diameter is multiplied by $\sqrt[3]{3}$, we shall have the diameter of another sphere whose volume is three times that of the given sphere; and so on.

But if the diameter is divided by $\sqrt[3]{2}$, we shall obtain the diameter of another sphere whose volume is only half as much as that of the given sphere; if it is divided by $\sqrt[3]{3}$, we shall have the diameter of another sphere whose volume is one-third that of the given sphere.

(*b*) Or, a process similar to that given in Note 2, *b*, Prob. XVIII., may be adopted.

Example 1.—Find the volume of a sphere whose diameter is 5 ft. 3 in.

Here, 5 ft. 3 in. $=63$ in.

Then, volume of sphere $=63^3 \times \frac{11}{21} = 130977$ cub. in. $=75$ cub. ft. 1377 cub. in.

Example 2.—Find the volume of a spherical shell whose outer diameter is 11 in. and the thickness of the shell is 1 in.

Here, the inner diameter $=$ outer diameter $-$ twice the thickness $=11-2=9$.

Then, by Note 3, *b*:

The cube of 11 is 1331, and the cube of 9 is 729; and the difference of these two is 602, which, being multiplied by $\frac{11}{21}$, gives $315\frac{1}{3}$ cub. in.—the volume of the shell.

Example 3.—The volume of a sphere is 4851 cub. in.; find its diameter.

By Note 1:

The diameter $=\sqrt[3]{4851 \times \frac{21}{11}}=\sqrt[3]{9261}=21$ in. $=1$ ft. 9 in.

$$\text{Formulæ}\left\{\begin{array}{lll} \text{I. Volume} & = & \text{diameter}^3 \times \frac{11}{21}. \\ \text{II. Surface} & = & \text{diameter}^2 \times \frac{22}{7}. \\ \text{III. Diameter} & = & \sqrt[3]{\text{volume} \times \frac{21}{11}}. \\ \text{IV. Diameter} & = & \sqrt{\text{surface} \times \frac{7}{22}}. \end{array}\right.$$

Examples.

Find the volume of a sphere, when its diameter is—

(1) Diameter 14 in. (2) Diameter 3 ft. 6 in.
(3) Diameter 5 ft. 10 in. (4) Diameter 10 ft. 6 in.
(5) Diameter 17 ft. 6 in. (6) Diameter 14·7 ft.
(7) Diameter 42 ft. (8) Diameter 84 ft.

Find the surface of a sphere whose diameter is—

(9) Diameter 10 in. (10) Diameter 1 ft. 9 in.
(11) Diameter 2 ft. 4 in. (12) Diameter 7 ft.
(13) Diameter 4·2 ft. (14) Diameter 10·5 ft.

Find the solid contents of a spherical shell whose external and internal diameters are, respectively—

(15) External diameter 10 in., and internal diameter 8 in.

(16) External diameter 14 in., and internal diameter 13 in.

(17) External diameter 3 ft., and internal diameter 2·8 ft.

(18) External diameter 1 ft. 7 in., and internal diameter 1 ft. 5 in.

Find the diameter of a sphere whose volume is—

(19) Volume 75 cub. ft. 1377 cub. in.

(20) Volume 179 cub. ft. 1152 cub. in.

(21) Volume 1047·816 cub. ft.

(22) Volume 38·808 cub. yds.

Find the diameter of a sphere, when its surface is—

(23) Surface 616 sq. in.

(24) Surface 38 sq. ft. 72 sq. in.

(25) Surface 427 sq. ft. 112 sq. in.

(26) Surface 9856 sq. ft.

(27) How many cubic inches of lead will be used in making a shell $\frac{1}{2}$ an in. thick, whose external diameter is 7 in.?

(28) How many gallons of water will a hemispherical bowl hold, whose diameter is 21 in., when a cubic foot contains $6\frac{1}{4}$ gallons?

(29) The weight of a cubic inch of iron is $4\frac{1}{2}$ oz. avoirdupois; how much will a solid ball weigh whose diameter is 5 in.?

(30) What must be the external diameter of a spherical shell $\frac{1}{2}$ an in. thick, that its hollow part may contain $113\frac{1}{7}$ cub. in.?

(31) A hemispherical bowl, made of lead 1 in. thick, holds 11 cub. ft. 396 cub. in. of water; how many cub. in. of lead were used in its construction?

(32) Find how many sq. feet of lead will be used in making 16 hemispherical bowls, when the diameter of each is 2 ft. 4 in.

(33) The circumference of the dome on a large building, in the shape of a hemisphere, is 66 ft.; find how many sq. ft. of lead will be used in covering it.

(34) Find the diameter of the mouth of a cannon that fires a ball of 24 lbs. weight, when 1 cub. in. of iron weighs [illegible]. avoirdupois.

(35**) What is the weight of a spherical shell 10 in. in diameter and 2 in. thick, composed of a substance of which 1 cub. ft. weighs 216 lbs.?

(36*) A circular room has perpendicular walls 15 ft. high, the diameter of the room being 28 ft.; the roof is a hemispherical dome. Find the cost of plastering the whole surface, at 9*d*. a sq. ft.; and the cost of a string moulding round the springing of the dome, at 15*d*. per ft.

(37*) The weight of an iron ball whose diameter is 4 in. is 9 lbs.; find the weight of a shell whose external and internal diameters are 8 in. and 5 in. respectively.

(38**) A shell is 7 in. in external diameter, and 2 in. thick; what weight of powder will fill it, if 30 cub. in. of powder weigh 1 lb.?

(39*) Find the weight of an 11-in. spherical shell, whose thickness is 3 in., made of iron, weighing 4 cwt. to the cubic foot.

(40**) A cylinder, 5 ft. long and 3 ft. in diameter, is closed by an hemisphere at each end; find the whole surface.

(41**) If an iron ball, 4 in. in diameter, weighs 9 lbs., what is the weight of a hollow iron shell, 2 in. thick, whose external diameter is 20 in.?

(42*) If 30 cub. in. of powder weigh 1 lb., what is the internal diameter of a shell that holds 15 lbs.?

(43*) If a leaden ball of 1 in. in diameter weighs $\frac{3}{14}$ lb., what is the diameter of a leaden ball that weighs 588 lbs.?

(44**) Supposing a cubic inch of iron to weigh 4·2 oz., find the weight of a spherical solid shot 6 in. in diameter; and prove that three spherical shots of 3 in., 4 in., and 5 in. respectively in diameter would make together the same weight.

(45*) How many gallons will a hemispherical bowl, 2 ft. 4 in. in diameter, hold (a gallon = 277 cub. in.)? and what would it cost to gild the inner surface, at $1\frac{1}{2}d$. per sq. in.?

(46*) A Stilton cheese is in the form of a cylinder, and a Dutch cheese in the form of a sphere. Determine the diameter of the Dutch cheese, which weighs 9 lbs., when a Stilton cheese, 14 in. high and 8 in. in diameter, weighs 12 lbs.

(47*) How many cubic inches of iron are there in a spherical cannon-ball 9 in. in diameter? If the ball is melted and cast into a conical mould, the base of which is 18 in. in diameter, find what the height of the cone will be.

(48**) A hemispherical basin, 15 ft. in diameter, will hold 120 times as much water as a cylindrical tub, the depth of which is 1 ft. 6 in.; find the diameter of the tub.

(49*) A pyramid of lead is 14 in. high, and stands on a square base 6 in. long on each side; how many spherical bullets ·75 in. in diameter can be made out of it?

(50**) How many square inches of gold-leaf will gild a globe 18 in. in diameter?

(51) Find the diameter of an iron ball whose volume is half a cubic foot.

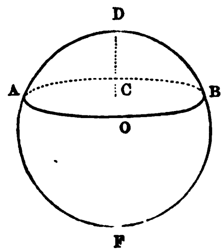
(52**) If 4 cub. ft. of stone weigh 9 cwt., find the weight of a stone hemisphere of radius 2 yds.

(53) The diameter of a globe is 7 in.; find the diameter of another globe whose volume is three times that of the former.

(54) The diameter of a sphere is 1 ft. 9 in.; find the diameter of another sphere whose volume is only $\frac{1}{8}$ that of the former.

XXVI. THE SEGMENT OF A SPHERE.

Definitions.—A segment of a sphere is any part of a sphere cut off by a plane.



Thus, ADB is a *segment* of a sphere; and also the remaining part of the sphere AFB is another segment.

If the plane passes through O, the centre of the sphere, then it will divide the sphere into two equal segments, called hemispheres.

The circle AB is called the *base* of the segment; and the perpendicular CD, drawn from C, the *centre* of the base AB, is the *height* of the segment ADB.

The *whole surface* of the segment of a sphere consists of the circular base AB and another portion called the curved or convex surface.

RULES.—(1) *To find the volume, when the radius of the base and the height of the segment are given.*

To three times the square of the radius of the base, add the square of the height; multiply the sum by the height, and this product by $\frac{11}{21}$.

(2) *To find the volume, when the diameter of the sphere and the height of the segment are given.*

From three times the diameter subtract twice the height of the segment; multiply the difference by the square of the height of the segment, and the product by $\frac{11}{21}$.

(3) *To find the curved or convex surface.*

Multiply the circumference of the sphere by the height of the segment.

(4) *To find the whole surface.*

To the curved surface (found by Rule 3), add the area of the circular base.

Example 1.—The diameter of the base of the segment of a sphere is 6 ft., and its height is 3 ft.; find its volume.

Now, radius of the base of the segment is $=\frac{6}{2}=3$ ft.; and three times the square of this $=3^2\times3=27$ ft.; and the square of the height $=3^2=9$ ft.

Then, volume $=(27+9)\times3\times\frac{11}{21}=36\times3\times\frac{11}{21}=56\frac{4}{7}$ cub. ft.

Example 2.—The height of the segment of a sphere is 6 in., and the diameter of the sphere is 16 in.; find the volume of the segment.

Now, three times the diameter $=3\times16$ in. $=48$ in.; and twice the height of the segment $=2\times6$ in. $=12$ in.

Then, volume $=(48-12)\times6^2\times\frac{11}{21}=36\times36\times\frac{11}{21}=678\frac{6}{7}$ cub. in.

Example 3.—Find the curved surface of the segment of a sphere whose height is 16 in., and the diameter of the sphere is 3 ft.

Now, the circumference of the sphere $=3$ ft. $\times\frac{22}{7}=36\times\frac{22}{7}$ $=\frac{792}{7}$ in.

Then, curved surface $=$ circumference of sphere $\times$ height of segment $=\frac{792}{7}\times16=\frac{12672}{7}$ sq. in. $=1810\frac{2}{7}$ sq. in. $=12$ sq. ft. $82\frac{2}{7}$ sq. in.

Formulæ:

I. Volume $=\{(3\times\text{radius of base}^2)+\text{height of segment}^2\}\times\text{height of segment}\times\frac{11}{21}$.

Volume $=(3\text{ diameter of sphere}-2\text{ height of segment})\times\text{height of segment}^2\times\frac{11}{21}$.

II. Curved surface $=$ circumference of sphere $\times$ height of segment.

III. Whole surface $=$ curved surface $+$ area of circular base.

EXAMPLES.

(1) The height of the segment of a sphere is 4 ft., and the diameter of the sphere is 29 ft.; find its volume.

(2) The radius of the base of the segment of a sphere is 24 in., and the height of the segment is 12 in.; find its volume.

(3) The radius of the base of the segment of a sphere is 40 ft., and its height is 20 ft.; find its volume.

(4) The radius of the base of the segment of a sphere is 16 in., and the radius of the sphere is 20 in.; find its volume.

(5) The diameter of a sphere is 21 ft., and the height of the segment is 5 ft.; find the curved surface.

(6) The inside of a washhand-basin is in the shape of the segment of a sphere; the distance across the top is 16 in., and its greatest depth is 6 in.; find how many pints of water it will hold, reckoning $6\frac{1}{4}$ gallons to the cubic foot.

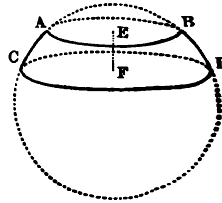
(7) The inside of a large bowl, in the shape of a segment of a sphere, measures 2 ft. 6 in. across at the top, and its greatest depth is 1 ft.; find how many gallons of water it will hold, when there are $6\frac{1}{4}$ gallons to the cubic foot.

(8) How many square feet of lead will be required for lining the inside of a bowl, in the shape of a segment of a sphere, which measures 40 in. across at the top, and whose greatest depth is 10 in.?

(9**) Find the weight of a pair of iron dumb-bells, each consisting of two spheres of $4\frac{1}{2}$ in. in diameter, joined by a cylindrical bar 6 in. long and 2 in. in diameter; an iron ball, 4 in. in diameter, weighing 9 lbs.

XXVII. THE ZONE OF A SPHERE.

Definitions.—A zone of a sphere is any part ABCD of a sphere which lies between two parallel planes AB and CD.



The upper and bottom surfaces, which are called the *ends* of a zone, are circular; and the straight line drawn from the centre of each circular end to its circumference is called the *radius* of that end.

The *height* EF of a zone is the *perpendicular distance* between the two parallel planes.

The *whole surface* of a zone will consist of the areas of the two circular ends and the area of the curved or convex portion lying between these circular ends.

RULES.—(1) *To find the volume of the zone of a sphere.*

To three times the sum of the squares of the radii of the two ends, add the square of the height; multiply the sum by the height, and the product by $\frac{11}{21}$.

(2) *To find the curved or convex surface.*

Multiply the circumference of the sphere by the height of the zone.

(3) *To find the whole surface.*

To the area of the curved surface (found by Rule 2), add the areas of the two circular ends.

Example 1.—Find the volume of the zone of a sphere, the radii of whose ends are, respectively, 8 in. and 11 in., and whose height is 4 in.

The square of the radius of one end$=8^2=64$; and the square of the radius of the other end$=11^2=121$.

Then, $3(64+121)+4^2=571$.

Therefore, volume$=571\times 4\times\frac{11}{21}=1196\frac{8}{21}$ cub. in.

Example 2.—Required the whole surface of the zone of a sphere, the perpendicular distance between the parallel ends being 3 ft., and the radii of the two ends being, respectively, 4 ft. and 6 ft.; and the diameter of the sphere is 16 ft.

The circumference of the sphere$=16\times\frac{22}{7}=\frac{352}{7}$.

Then,

The curved surface $=\frac{352}{7}\times 3=\frac{1056}{7}$ sq. ft.	$=150\frac{6}{7}$ sq. ft.
Area of upper end$=8^2\times\frac{11}{14}=\frac{704}{14}$ sq. ft.	$=50\frac{2}{7}$ sq. ft.
Area of lower end$=11^2\times\frac{11}{14}=\frac{1331}{14}$ sq. ft.	$=95\frac{1}{14}$ sq. ft.
The whole surface	$=296\frac{3}{14}$ sq. ft.

Formulæ:
- I. Volume$=\{3(\text{AE}^2+\text{FC}^2)+\text{EF}^2\}\times\text{EF}\times\frac{11}{21}$.
- II. Curved surface $=$ circumference of sphere $\times$ height of zone.
- III. Whole surface $=$ curved surface $+$ areas of two circular ends.

Examples.

(1) Find the volume of the zone of a sphere, the radii of whose ends are 12 in. and 10 in. respectively, and the height of the zone is 6 in.

(2) The radii of the ends of the zone of a sphere are 6 ft. and 8 ft., and the height is 3 ft.; find the volume.

(3) Find the volume of the middle zone of a sphere, whose top and bottom diameters are each 8 ft., and the height is 4 ft.

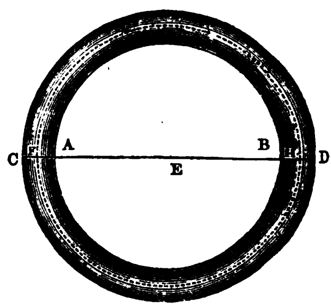
(4) Find the whole surface of the middle zone of a sphere whose diameter at the top and bottom is 9 ft. 4 in., and the height of the zone is 5 ft. 6 in.

(5) A bowl is in the shape of a zone of a sphere; the radius of the top is 16 in. and of the bottom 4 in.; the greatest depth is 10 in. Find the number of cubic feet of water that it will hold.

(6) A bowl is in the shape of a zone of a sphere; it measures 2 ft. across at the top, and 10 in. across at the bottom; and its greatest depth is 13 in. Find the number of gallons of water that it will hold, when 1 cubic foot contains $6\frac{1}{4}$ gallons.

XXVIII. THE CIRCULAR RING.

Definitions.—A circular ring may be defined to be a cylinder bent into a ring.



The *length* of a circular ring is represented by the dotted line, and is equal to the circumference of a circle of which FH or (AB + thickness of ring) is the diameter.

The *length* of the ring is also equal to half the sum of the outer and inner circumferences.

The *thickness* of the ring AC or BD = half the difference of the outer and inner diameters.

The *outer* diameter = *inner* diameter + twice the thickness of the ring.

The *inner* diameter = *outer* diameter − twice the thickness of the ring.

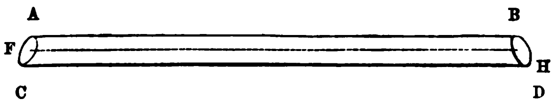
RULES.—(1) *To find the volume of a circular ring.*

Multiply the area of a circular section of the ring (which is found by multiplying the square of the thickness of the ring by $\frac{11}{14}$) by the length of the ring.

(2) *To find the area of the surface of a circular ring.*

Multiply the circumference of a circular section of the ring (which is found by multiplying the thickness of the ring by $\frac{22}{7}$) by the length of the ring.

Note.—A circular ring may be converted into a straight rod, and its volume will then be found by the rule given for finding the volume of a cylinder.



The length of the cylinder is $\text{FH}=\frac{\text{AB}+\text{CD}}{2}$=half the sum of the outer and inner circumferences.

Example 1.—The inner diameter of a ring is 7 in., and its thickness is 1 in.; find its volume.

The outer diameter $=7+2=9$ in.

The area of a circular section of the ring $=1^2\times\frac{11}{14}$ $=\frac{11}{14}$ sq. in.

The inner boundary of the ring $=7\times\frac{22}{7}=22$ in.; and outer boundary $=9\times\frac{22}{7}=\frac{198}{7}$ in.; therefore the length of the ring $=\frac{1}{2}(22+\frac{198}{7})=25\frac{1}{7}$ in.

Hence, volume of ring = area of circular section × length $=\frac{11}{14}\times25\frac{1}{7}=\frac{968}{49}=19\frac{37}{49}$ cub. in.

Example 2.—The outer diameter of a ring is 10 in., and the inner diameter is 8 in.; find its volume.

Now, the thickness of the ring $=\frac{10-8}{2}=1$ in.

The area of a circular section of the ring $=1^2\times\frac{11}{14}$ $=\frac{11}{14}$ sq. in.

The length of the ring is the circumference of a circle whose diameter is FH, or $\overline{\text{AB}+\text{thickness}}$ (see figure); that is, $=9\times\frac{22}{7}=\frac{198}{7}$ in.

Then, volume of ring = area of circular section × length $=\frac{11}{14}\times\frac{198}{7}=\frac{1089}{49}=22\frac{11}{49}$ cub. in.

Example 3.—The inner diameter of a ring is 12 in., and its thickness is 2 in.; find the area of its surface.

The thickness of the ring is 2 in.; therefore the circumference of a circular section of it $=2\times\frac{22}{7}=\frac{44}{7}$ in.

And the length of the ring is the circumference of a circle whose diameter is FH, or $\overline{AB+\text{thickness}}$ of ring, or 14 in.; and is therefore $=14\times\frac{22}{7}=44$ in.

Hence, surface of ring = circumference of circular section × length of ring

$$=\frac{44}{7}\times 44=\frac{1936}{7}\text{ sq. in.}=276\tfrac{4}{7}\text{ sq. in.}$$

Formulæ:
- I. Volume = area of circular section × length.
- II. Surface = circumference of circular section × length.

Examples.

(1) The inner diameter of an iron ring is 12 in., and its thickness is 3 in.; find how many cubic inches of iron were used in its construction.

(2) The inner diameter of a ring is 16 in., and its thickness is 2 in.; find its volume.

(3) The inner diameter of a ring is 21 in., and its outer diameter is 24 in.; find its volume.

(4) The inner diameter of an iron ring is 14 in., and its thickness is 3 in.; find its weight, when 1 cubic inch of iron weighs $4\frac{1}{2}$ oz. avoirdupois.

(5) Find the cost of painting a circular ring whose outer diameter is 16 ft., and inner diameter is 14 ft., at 3*d.* per sq. ft.

(6) Iron weighs $4\frac{1}{2}$ oz. avoirdupois per cubic inch; find the weight of an iron ring 2 ft. thick, and whose outer diameter is 15 ft.

XXIX. IRREGULAR SOLIDS.

Definition.—An irregular solid is one which is irregular in shape, and is not included under any of the previous definitions—such as many blocks of stone, pieces of iron, &c.

RULES.—(1) *To find the volume of an irregular solid, which is heavier than water, by placing it in a vessel of known capacity.*

(*a*) Put the solid into any vessel of convenient form, such as a rectangular cistern or cylinder. Pour into the vessel as much water as will *quite* cover the solid. Take out the solid, and notice how much the water sinks in consequence. Then the volume of the solid is equal to the volume of a column of water having for its *base* the base of the vessel, and for its *height* the distance through which the water has sunk by removing the solid.

(*b*) Or, fill the vessel full of water at the first; then put in the solid gently, and measure the volume of the water that runs over the sides of the vessel. (The volume of the water running over = volume of the solid.)

(2) *To find the volume of an irregular solid from its weight.*

Divide the weight of the solid, brought into ounces, by the weight of a cubic inch of that particular substance; and the quotient is the number of cubic inches in the solid.

(3) *To find approximately the volume of an irregular solid by different measurements.*

Take breadth and depth of the solid at different points *equally* distant from each other. Take the

average of these for the *mean breadth* and *mean depth.* Then the volume of solid=mean breadth $\times$ mean depth $\times$ length.

(4) *To find the capacity of any vessel.*

Weigh the vessel when *empty*, and then weigh it *full of water*; the difference of these two weights is the weight of the water in the vessel. Divide the weight of water, brought into ounces, by 1000 oz.; and the quotient is the volume of the vessel in cubic feet.

EXAMPLES.

(1) The length of a cubical vessel is 3 ft.; an irregularly-shaped stone is put into it, and water poured into the vessel until the solid is quite covered: in taking out the stone, the water sinks 7 in. Find the volume of the stone.

(2) In a cylindrical vessel, whose diameter is 14 in., is placed an irregular piece of iron; water is then poured in until the solid is covered: on taking out the iron, it is found that the water in the vessel has sunk $1\frac{1}{2}$ in. Find the volume of the iron.

(3) An irregular piece of granite is placed in a rectangular vessel whose length is 4 ft. 2 in. and breadth 3 ft.; water is poured into the vessel until the stone is covered: when the stone is taken out, the water sinks 8 in. Find the volume of the stone.

(4) When a body entirely immersed in water is taken out of a vessel, it is found that it will require $2\frac{1}{4}$ gallons *more* to raise the water in the vessel to the same level as it was before; find the volume of the solid, when 1 cubic foot contains $6\frac{1}{5}$ gallons of water.

(5) Into a vessel, which is full of water, is placed a block of marble; it is found that $3\frac{1}{2}$ gallons of water have run over the sides of the vessel. Find the volume of the marble, when 1 cubic foot contains $6\frac{1}{5}$ gallons of water.

(6) A certain vessel when empty weighs 50 lbs.; but when filled with water, it weighs $331\frac{1}{4}$ lbs. Find the capacity of the vessel, when 1 cubic foot of water weighs 1000 oz. avoirdupois.

(7) An iron ring weighs $337\frac{1}{2}$ lbs.; find how many cubic inches there are in the ring, when 1 cub. in. of iron weighs $4\frac{1}{2}$ oz. avoirdupois.

(8) A piece of lead weighs $123\frac{3}{4}$ lbs.; find its volume, when 1 cub. in. of lead weighs $6\frac{3}{5}$ oz. avoirdupois.

(9) A brass cannon weighs $3\frac{1}{2}$ tons; find how many cubic inches of brass were used in its construction, when 1 cub. ft. of water weighs 1000 oz. avoirdupois, and brass is eight times as heavy as water.

(10) How many cubic feet are there in a ton of coal, when 1 cub. ft. of water weighs 1000 oz. avoirdupois, and coal is $1\frac{1}{4}$ times as heavy as water?

(11) Find approximately the solid contents of a block of marble, whose four equidistant measurements are 6 ft. by 5 ft., 5 ft. by 4 ft., 4 ft. by 3 ft., 3 ft. by 2 ft.; the length of the block is 16 ft.

(23) 131·521 in. = 10 ft. 11·521 in. (24) £34 13*s.* 9*d.*
(25) 133956 sq. yds. = 27 ac. 2 roods 28 poles 9 sq. yds.
(26) 16 yds.
(27) 29241 sq. yds. = 6 ac. 0 rood 6 poles 19½ sq. yds.
(28) 4 in. (29) £350. (30) £425.
(31) 7·789 yds. wide. (32) 70 yds. 2 ft.

III. THE RECTANGLE.

(1) 6 sq. ft. 135 sq. in. (2) 14 sq. yds. 6 sq. ft.
(3) 99 sq. ft. 95 sq. in. (4) 49 sq. ft. 54 sq. in.
(5) 8 sq. ft. 27¾ sq. in. (6) 11 sq. yds. 7 sq. ft. 48 sq. in.
(7) 37800 sq. yds. = 7 ac. 3 roods 9 poles 17¾ sq. yds.
(8) 226200 sq. links = 2 ac. 1 rood 1 pole 27·83 sq. yds.
(9) 7705 sq. yds. = 1 ac. 2 roods 14 poles 21½ sq. yds.
(10) 8748 sq. yds. (11) 41772 sq. yds.
(12) 33·966 sq. ft. (13) 11 yds. 2 ft.
(14) 9 yds. 2 ft. (15) 4 yds. 2 ft. 8 in.
(16) 121 yds. (17) 127 yds.
(18) 2 chains 50 links. (19) 11 yds.
(20) £5 6*s.* 3¾*d.* (21) 7168 tiles.
(22) 6 yds. 2 ft. 7½ in. (23) 23 yds. 2 ft. 1⅓ in.
(24) £234 1*s.* (25) £4 2*s.* 2⅔*d.*
(26) 1 ft. 8 in. (27) £4166 13*s.* 4*d.*
(28) 6 ft. (29) £35 12*s.*; £32.
(30) 151¼ yds. (31) 7 chains.
(32) Square = 625 sq. ft.; rectangle = 400 sq. ft.; difference 225 sq. ft.
(33) 71$\frac{3}{7}$ links. (34) 56 yds.
(35) £23 0*s.* 4$\frac{11}{36}$*d.* (36) £37 16*s.* 4½*d.*
(37) £12 19*s.* 9$\frac{3}{16}$*d.* (38) 15 ft.
(39) Square = 9604 sq. yds.; rectangle = 2548 sq. yds.; difference = 7056 sq. yds.
(40) 82668 sq. yds. = 17 ac. 0 rood 12 poles 25 sq. yds.
(41) 97 yds. 2½ ft. (42) 28⅓ planks.
(43) £11 5*s.* (44) 2 ft. 4 in.
(45) 14 ft. (46) 50 yds.
(47) 70 yds.
(48) Diagonal of rectangle = 53 yds.; diag. of square = 50·199 yds.
(49) £118 8*s.* (50) 73$\frac{2}{27}$ sq. yds.
(51) £28 4*s.* (52) 10½ in.
£3 14*s.* 3*d.*

IV. THE OBLIQUE PARALLELOGRAM.

(1) 3 sq. ft. 93 sq. in. (2) 1164 sq. ft. 54 sq. in.
(3) 161 sq. yds. 8 sq. ft.
(4) 25200 sq. yds. = 5 ac. 0 rood 33 poles $1\frac{3}{4}$ sq. yds.
(5) 1250 sq. poles = 7 ac. 3 roods 10 poles.
(6) 337500 sq. links = 3 ac. 1 rood 20 poles.
(7) 5 ft. 6 in. (8) 143 yds.
(9) 6 chains 25 links. (10) £45 14*s.* $4\frac{1}{2}$*d.*
(11) 4 chains 20 links. (12) 6560 sq. ft.
(13) 3960 sq. yds. (14) £90.
(15) 1629·12 sq. ft. (16) 15·874 ft.
(17) Side = 75 yds.; height = 72 yds.

V. THE TRIANGLE.

(1) 500. (2) 105 sq. ft. 132 sq. in. (3) 78·3 sq. yds.
(4) 430560 sq. links = 4 ac. 1 rood 8 poles 27·328 sq. yds.
(5) 12375 sq. yds. = 2 ac. 2 roods 9 poles $2\frac{3}{4}$ sq. yds.
(6) 20 ft. (7) 23 ft. 8 in. (8) 8 chains 33 links.
(9) 5 chains 14 links. (10) 600.
(11) 2730 sq. ft. (12) 5460 sq. ft.
(13) 22386 sq. yds. (14) 92·4 sq. yds.
(15) 7140 sq. yds. (16) 204·6 sq. yds.
(17) £250 7*s.* 8*d.* (18) 420 sq. ft.
(19) 4876 sq. yds. 6 sq. ft. (20) £1 10*s.*
(21) Square = 225 sq. ft.; triangle = 173·205 sq. ft.
(22) 1444 sq. yds.
(23) 46410 sq. links = 1 rood 34 poles 7·744 sq. yds.
(24) £156 3*s.* $11\frac{73}{121}$*d.*
(25) Parts 32 ft., 18 ft.; areas 216 sq. ft., 384 sq. ft.
(26) 393 sq. ft. 48 sq. in. (27) 16·8 ft.
(28) AC = $14\frac{1}{12}$ ft.; AB = $21\frac{2}{3}$ ft.; BD = $22\frac{3}{4}$ ft.; and area = $147\frac{7}{8}$ sq. ft.
(29) 23·61 ft.
(30) Triangle = 10000 sq. yds., or 2 ac. 10 poles $17\frac{1}{2}$ yds.; parts = 2000 sq. yds., or 1 rood 26 poles $3\frac{1}{2}$ sq. yds.; and 8000 sq. yds.. or 1 ac. 2 roods 24 poles 14 sq. yds.
(31) 7·154 ft.
(32) Area = 504 sq. ft.; perpendicular = 21 ft.
(33) £1179 15*s.* (34) 420 sq. ft.
(35) 62·161 yds. (36) 800 sq. ft. (37) 2401 sq. ft.
(38) 1385·64 sq. ft. (39) 779·4 sq. ft.

VI. THE TRAPEZIUM.

(1) 960 sq. ft. (2) 586·5 sq. ft.
(3) 40000 sq. yds., or 8 ac. 1 rood 2 poles 9½ sq. yds.
(4) 258600 sq. links, or 2 ac. 2 roods 13 poles 22·99 sq. yds.
(5) 54 ft. (6) 248 yds.
(7) 10 chains 20 links. (8) £87.
(9) 60 ft. 9 in. (10) 4896 sq. ft.
(11) 41216 sq. ft. (12) 2895·955 sq. ft.
(13) 12585·706 sq. ft. (14) 463·7408 sq. ft.
(15) 126 sq. yds. (16) 1260 sq. chains, or 126 ac.
(17) 726531 sq. links, or 7 ac. 1 rood 2·4496 poles.

VII. THE TRAPEZOID.

(1) 1050 sq. ft. (2) 250 sq. ft.
(3) 17000 sq. yds., or 3 ac. 2 roods 1 pole 29¾ sq. yds.
(4) 1040000 sq. links, or 10 ac. 1 rood 24 poles.
(5) 23 yds. 1 ft. (6) 5 chains 14 links.
(7) £6 16*s*. 6*d*. (8) 3 chains 50 links.
(9) 248 yds.
(10) 148200 sq. lin' s, or 1 ac. 1 rood 37 poles 3·63 sq. yds.
(11) £2 12*s*. per acre.

VIII. THE REGULAR POLYGON.

(1) 389·7 sq. in., or 2 sq. ft. 101·7 sq. in.
(2) 9353·16 sq. ft. (3) 3270·51 sq. ft.
(4) 4345·56 sq. ft.
(5) 49242·88 sq. links, or 1 rood 38 poles 23·855392 sq. yds.
(6) 1453·48 sq. ft. (7) £45. (8) 1000 tiles.
(9) Hexagon 16627·84 sq. ft.; octagon 17382·24 sq. ft.

IX. THE IRREGULAR POLYGON.

(1) 725010 sq. links, or 7 ac. 1 rood 12·584 sq. yds.
(2) ABD + BCD = 4368 + 4620 = 8988 sq. yds., or 1 ac. 3 roods 17 poles 3¾ sq. yds.
(3) ABE + EBD + BCD = 2070 + 4200 + 966 = 7236 sq. yds., or 1 ac. 1 rood 39 poles 6¼ sq. yds.
(4) ABCE + EDC = 6264 + 1680 = 7944 sq. yds., or 1 ac 2 roods 22 poles 18½ sq. yds.

(5) ABC+ACE+CDE+AFE = 3192 + 6006 + 4290 + 714 = 14202 sq. yds., or 2 ac. 3 roods 29 poles $14\frac{3}{4}$ sq. yds.

(6) AFHE+AFB+BDH+BCD = 30625 + 14000 + 2475 + 38000 = 85100 sq. yds., or 17 ac. 2 roods 13 poles $6\frac{3}{4}$ sq. yds.

(7) AGHF+AGB+FHE+ BCE+CDE = 12540 + 1701 + 1200 + 19380 + 17940 = 52761 sq. yds., or 10 ac. 3 roods 24 poles 5 sq. yds.

(8) AGDB+BCD+GFED = 9010 + 1066 + 6880 = 16956 sq. yds., or 3 ac. 2 roods 16 sq. yds.

X. OFFSETS.

(1) 1044 sq. ft.
(2) 464 sq. yds., or 15 poles $10\frac{1}{4}$ sq. yds.
(3) 2942 sq. yds., or 2 roods 17 poles $7\frac{3}{4}$ sq. yds.
(4) 2145 sq. yds., or 1 rood 30 poles $27\frac{1}{2}$ sq. yds.
(5) 34020 sq. links, or 1 rood 14 poles 13·068 sq. yds.
(6) 4860 sq. yds., or 1 ac. 20 sq. yds.

XI. THE CIRCLE.—THE CIRCUMFERENCE AND DIAMETER.

(1) 1 ft. 10 in.
(2) 23 ft. 10 in.
(3) 256 ft. 8 in.
(4) 160 yds. 2 ft. 2 in.
(5) 4 fur. 16 poles.
(6) 29 chains 4 links.
(7) 111·54 ft.
(8) 374·22 ft.
(9) 4 ft. 1 in.
(10) 11 ft. 8 in.
(11) 19 yds. 1 ft. 4 in.
(12) 2 fur. 4 poles.
(13) 1 chain 5 links.
(14) 12·25 ft.
(15) 99·96 ft.
(16) 120 revolutions.
(17) 3 ft. $9\frac{9}{11}$ in.
(18) $3\frac{1}{2}$ in. wide.
(19) $17\frac{1}{2}$ ft. wide.
(20) 63 yds.
(21) 73 yds. 1 ft.
(22) 11 ft. $9\frac{3}{7}$ in.
(23) 15 chains $90\frac{10}{11}$ links.
(24) 2 ft. $7\frac{3}{7}$ in.

XII. THE AREA OF A CIRCLE.

(1) 1 sq. ft. 10 sq. in.
(2) 180 sq. ft. 106 sq. in.
(3) 386 sq. ft. 10 sq. in.
(4) 17 sq. yds. 1 sq. ft.
(5) 186·34 sq. yds.
(6) 273 sq. yds. 7 sq. ft.
(7) 81466 sq. links, or 8 sq. chains 1466 sq. links.
(8) 75·46 sq. ft.
(9) 4 sq. ft. 40 sq. in.

(10) 154 sq. ft. (11) 17 sq. ft. 16 sq. in.
(12) 5544 sq. yds.
(13) 985600 sq. yds., or 203 ac. 2 roods 21 poles $24\frac{3}{4}$ sq. yds.
(14) 2 ft. 4 in. (15) 37 yds. 1 ft. (16) 56 yds.
(17) 98 yds. (18) 192·249 yds. (19) 22 ft.
(20) 396 yds. (21) 11 chains 44 links. (22) 352 yds.
(23) 308 sq. ft. (24) 4 sq. yds. 2 sq. ft. 72 sq. in.
(25) 49·638 ft. (26) 154 sq. ft. (27) £136 8*s.*
(28) £22 9*s.* $5\frac{1}{7}d.$ (29) 2·984 yds. (30) 39·597 ft.
(31) 1050 sq. in. (32) 28·284 in. (33) 4 sq. ft. 96 sq. in.
(34) 12·409 ft. (35) 6·2048 ft. (36) 5544 sq. ft.
(37) £18 17*s.* $1\frac{5}{7}d.$ (38) $754\frac{2}{7}$ sq. ft.
(39) Bath 240 sq. ft. 90 sq. in.; space left 65 sq. ft. 90 sq. in.
(40) Square 1089 sq. ft.; circle 1386 sq. ft.; triangle 838·312 sq. ft.
(41) $116\frac{2}{7}$ sq. yds. = £1 18*s.* $9\frac{1}{7}d.$ (42) $696\frac{3}{7}$ sq. ft. = £78 7*s.* $8\frac{1}{4}d.$
(43) $539\frac{3}{8}$ sq. ft. (44) 5·196 ft.; and 7·348 ft.
(45) £1 11*s.* $9\frac{17}{28}d.$ (46) 792 yds. = £69 6*s.*
(47) Diameter 40·414 yds.; widths 8·37 yds. and 6·423 yds.
(48) 20·296 yds.
(49) $3444\frac{4}{7}$ sq. ft. = £12 15*s.* $1\frac{53}{63}d.$; $282\frac{6}{7}$ yds. = £3 10*s.* $8\frac{4}{7}d.$; and total = £16 5*s.* $10\frac{26}{63}d.$
(50) $36927\frac{11}{14}$ sq. ft. = £512 17*s.* $8\frac{13}{21}d.$

XIII. THE CHORDS OF A CIRCLE.

(1) $41\frac{2}{3}$ ft. (2) $41\frac{2}{7}$ ft. (3) 50 ft. $4\frac{23}{25}$ in.
(4) 127·5 ft. (5) 5·625 ft. (6) 4 ft.
(7) 14 ft. (8) 28·8 ft. (9) 1 ft. 3 in.
(10) 5 ft. 5 in. (11) 7·7 ft. (12) 1·75 ft.
(13) 2 ft. 1 in. (14) 5 ft. 5 in. (15) 69·375 ft.
(16) 102 ft. (17) 240 ft.

XIV. THE ARC OF A CIRCLE.

1) 11 in. (2) 11 in. (3) 45°.
(4) $57\frac{3}{11}$°. (5) 14 ft. $7\frac{1}{3}$ in. (6) 4 ft. $4\frac{1}{2}$ in.
(7) 42 ft. (8) $16\frac{1}{2}$ ft. (9) 33·312 in.
(10) 27·776 in. (11) $85\frac{1}{3}$ ft. (12) 141·209 ft.
(13) Less arc = 107·274 ft.; greater arc = 225·868 ft.
(14) Diameter = 289 ft.; arc = $282\frac{2}{3}$ ft.
(15) 222·562 ft. (16) 82·81 ft. (17) 246·336 ft.

XV. THE SECTOR OF A CIRCLE.

(1) 77 sq. in. (2) 1 sq. ft. $112\frac{2}{3}$ sq. in. (3) 616 sq. ft.

(4) Radius = 35 ft.; perimeter = 92 ft.

(5) 17 ft. 6 in. (6) 402·025 sq. ft. (7) 578·6625 sq. ft.

(8) 903 sq. ft. (9) 159·8842 sq. ft. (10) 1563·129 sq. ft.

XVI. THE SEGMENT OF A CIRCLE.

(1) 546·175 sq. ft. (2) 191·482 sq. ft. (3) $159\frac{3}{16}$ sq. ft.

(4) $442\frac{3}{16}$ sq. ft. (5) $392\frac{10}{21}$ sq. ft. (6) $342\frac{2}{15}$ sq. ft.

(7) $816\frac{2}{3}$ sq. ft. (8) $2066\frac{2}{3}$ sq. ft.

XVII. THE ELLIPSE.

(1) 11 ft. (2) 33 ft. (3) 190 yds. 2 ft.

(4) 11 chains. (5) 61 ft. (6) 100 ft.

(7) 13 ft. 8 in. (8) 115 yds. (9) 8470 sq. ft.

(10) 935 sq. ft. 110 sq. in. (11) 1078 sq. yds.

(12) 7528 sq. yds. 8 sq. ft. (13) 3 sq. chains 7400 sq. links.

(14) 20 ft. (15) 3 ft. 4 in. (16) 196 yds.

(17) 2 chains 50 links. (18) £29 15*s.* 0*d.* (19) 112 yds.

(20) 62 yds. (21) £2 2*s.* $5\frac{1}{7}$*d.*

XVIII. THE CUBE.

(1) V. = 27 cub. in.; S. = 54 sq. in.

(2) V. = 1 cub. ft. 1016 cub. in.; S. = 8 sq. ft. 24 sq. in.

(3) V. = 614·125 cub. in.; S. = 3 sq. ft. $1\frac{1}{2}$ sq. in.

(4) V. = 11 cub. ft. 675 cub. in; S. = 30 sq. ft. 54 sq. in.

(5) V. = 614 cub. ft. 216 cub. in.; S. = 433 sq. ft. 72 sq. in.

(6) V. = 8000 cub. ft.; S. = 2400 sq. ft.

(7) V. = 381 cub. ft. 135 cub. in.; S. = 315 sq. ft. 54 sq. in.

(8) V. = 5359 cub. ft. 648 cub. in.; S. = 1837 sq. ft. 72 sq. in.

(9) V. = 7226 cub. ft. 640 cub. in.; S. = 2242 sq. ft. 96 sq. in.

(10) 7 in. (11) 1 ft. 5 in. (12) 2 ft. 6 in.

(13) 4 ft. 1 in. (14) 17 ft. (15) 8·4 ft.

(16) 6 tons 2 qrs. 14 lbs. (17) 405 sq. ft.

(18) 6·299 ft. (Note 2, b). (19) 65·208 in.

(20) 9 tons 8 cwt. 1 qr. 9 lbs. 12 oz.

(21) 4·473 in. (22) 512 cub. ft.
(23) 6·868 ft. (Note 2, b); 495 cub. ft. too large.
(24) 3375 cub. ft. (25) 3 ft. 6 in.
(26) 281 sq. ft. 36 sq. in. (27) 16 tons 5 cwt. 1 qr. 22 lbs.
(28) 5 ft. (29) 2·616 ft.

XIX. THE RECTANGULAR PARALLELOPIPED.

(1) V. = 480 cub. in.; S. = 2 sq. ft. 88 sq. in.
(2) V. = 7 cub. ft. 1344 cub. in.; S. = 23 sq. ft. 112 sq. in.
(3) V. = 53 cub. ft. 1152 cub. in.; S. = 86 sq. ft. 112 sq. in.
(4) V. = 2560 cub. ft.; S. = 1216 sq. ft.
(5) V. = 2703 cub. ft. 1620 cub. in.; S. = 1177 sq. ft. 864 sq. in.
(6) V. = 157·78125 cub. ft.; S. = 178·375 sq. ft.
(7) 970 cub. in. (8) 3 cub. ft. 1596 cub. in.
(9) 59 cub. ft. 360 cub. in. (10) 6 in.
(11) 10 in. (12) 1 ft. 2 in.
(13) 8 ft. 8 in. (14) 4 ft. 6 in.
(15) £63 7*s*. 11$\frac{35}{48}$*d*. (16) 3229 cub. ft. 1044 cub. in.
(17) 1 ton 10 lbs. (18) 18 cwt. 15 lbs. 4 oz.
(19) 62 sq. ft. 24 sq. in. (20) 15360 bricks.
(21) 3 in. (22) 1080 books.
(23) 1 ft. 8 in. (24) 46$\frac{1}{2}$ cub. ft.
(25) 30000 bricks. (26) 9 cub. ft. 368 cub. in.
(27) £12 6*s*. 5$\frac{1}{2}$*d*. (28) 3 tons 4 cwt. 3 qrs. 4 lbs. 13 oz.
(29) 24 ft. (30) 38·51 in.
(31) 1 ton 3 cwt. 3 qrs. 22 lbs. 4$\frac{2}{3}$ oz.
(32) 160$\frac{230}{277}$ gallons; 14 cwt. 1 qr. 15 lbs. 5$\frac{1}{4}$ oz.
(33) 1 cub. ft. 270 cub. in. (34) ·000004 in.
(35) 1815 cub. ft.
(36) Length = 16 ft.; breadth = 12 ft.; depth = 8 ft.
(37) Length = 7·942 ft.; breadth = 6·354 ft.; depth = 4·765 ft.
(38) 496 gallons.

XX. THE CYLINDER AND THE PRISM.

(1) 804 cub. in. (2) 4 cub. ft. 942 cub. in.
(3) 126 cub. ft. (4) 126·9 cub. in.
(5) 19 cub. ft. 168 cub. in. (6) 180 cub. ft.
(7) V. = 1540 cub. in.; S. = 3 sq. ft. 8 sq. in.
V. = 21 cub. ft. 672 cub. in.; S. = 36 sq. ft. 96 sq. in.

(9) V. = 235 cub. ft. 1404 cub. in.; S. = 115 sq. ft. 72 sq. in.
(10) V. = 821 cub. ft. 576 cub. in.; S. = 352 sq. ft.
(11) V. = 42 cub. ft. 1344 cub. in.; S. = 73 sq. ft. 48 sq. in.
(12) 336 cub. in. (13) 720 cub. ft.
(14) 1344 cub. ft. (15) 7 in.
(16) 1 ft. 2 in. (17) 3 ft. 6 in.
(18) 2 ft. 6 in. (19) 8 ft.
(20) 20 ft. (21) 31 cub. ft. 864 cub. in.
(22) 311·76 cub. ft. (23) 2160 cub. ft.
(24) 31177·2 cub. in., or 18 cub. ft. 73·2 cub. in.
(25) 1920 cub. ft. (26) 1996 sq. ft. 72 sq. in.
(27) $7\frac{6}{7}$ cub. yds. (28) £7 2*s.* $7\frac{1}{9}$*d.*
(29) £3 2*s.* $10\frac{2}{7}$*d.* (30) $1746\frac{2}{3}$ cub. yds.
(31) 9*s.* 4*d.* per cub. yd. (32) 7392 yds.
(33) 32 cub. ft. 144 cub. in. (34) £7 8*s.* $4\frac{15}{112}$*d.*
(35) 1496 cub. ft. (36) 3 cub. ft. 360 cub. in.; 19*s.* 3*d.*
(37) 187·0632 cub. in. (38) £19 17*s.* 10*d.*; £34 9*s.* 4*d.*
(39) 2 cwt. 2 qrs. 14 lbs. $10\frac{2}{7}$ oz. (40) 2387 gallons.
(41) 1 cub. ft. $1157\frac{1}{7}$ cub. in. (42) 96 cub. ft. 432 cub. in.
(43) 8 ft. (44) $8288\frac{7}{36}$ gallons.
(45) £18 6*s.* 8*d.*; £1 8*s.* $6\frac{2}{9}$*d.* (46) 1 ft. $3\frac{45}{77}$ in.
(47) Price of wood = £1 1*s.* $2\frac{4}{7}$*d.*; workmanship = £1 13*s.* $11\frac{11}{35}$*d.*
(48) $117333333\frac{1}{3}$ cub. yds. (49) $488\frac{8}{11}$ coins.
(50) 198·886 in., or 16 ft. 6·886 in. (51) $8\frac{1}{4}$ lbs.
(52) 96 cub. ft. $740\frac{4}{7}$ cub. in. (53) $342\frac{2}{9}$ revolutions.

XXI. THE PYRAMID AND THE CONE.

(1) 1260 cub. in. (2) 8 cub. ft. 576 cub. in.
(3) 67 cub. ft. (4) 1 cub. ft. 428 cub. in.
(5) 128 cub. ft. 576 cub. in. (6) 114·986 cub. ft.
(7) 51 cub. ft. 576 cub. in. (8) 479 cub. ft. 192 cub. in.
(9) V. = 59 cub. ft. 1536 cub. in.; S. = 64 sq. ft. 24 sq. in.
(10) V. = 21 cub. ft. 672 cub. in.; S. = 32 sq. ft. 56 sq. in.
(11) V. = 5280 cub. ft.; S. = 1395 sq. ft. $61\frac{5}{7}$ sq. in.
(12) V. = 1232 cub. ft.; S. = 550 sq. ft.
(13) 448 cub. in. (14) 4 cub. ft. 864 cub. in.
(15) 33 cub. ft. 576 cub. in. (16) 34·64 cub. ft.
(17) 935·316 cub. ft. (18) 384 sq. ft.
(19) 177·588 sq. ft. (20) 5096 sq. ft.
(21) 4 ft. 8 in. (22) 11·2 ft.

(23) 6 ft. 6 in. (24) 10 ft. (25) 12 ft.
(26) 28 ft. (27) 51710·933 cub. ft.
(28) 266·29 sq. ft. (29) 185 sq. ft.
(30) 103488 cub. yds. (31) 18·935 ft.
(32) 3 cub. ft. 1116 cub. in. (33) 198 oz. avoirdupois.
(34) 35·332 cub. ft. (35) £73 2*s.* 6*d.*
(36) $77\frac{21}{22}$ cub. ft., or 77 cub. ft. $1649\frac{5}{11}$ cub. in.
(37) 96·052 sq. ft. (38) 34 cub. ft. $310\frac{1}{2}$ cub. in.
(39) 38 sq. ft. 72 sq. in. (40) V. $=47\frac{1}{7}$ cub. in.; S. $=47\frac{1}{7}$ sq. ft.
(41) 744 sq. ft. (42) £13 19*s.* $8\frac{4}{5}$*d.*
(43) $116\frac{37}{154}$ cub. ft. (44) $1880\frac{3704}{7763}$ gallons.
(45) 33 cwt. 3 qrs. (46) $27\frac{59}{63}$ yds.
(47) $326\frac{6}{7}$ yds. (48) $3\frac{1}{7}$ cub. in.
(49) 21·96 yds. of canvas; $209\frac{11}{21}$ cub. ft.
(50) $\frac{32}{1701}$ cub. in. (51) 22·956 ft.
(52) £11 0*s.* 6*d.*; £9 2*s.* $8\frac{2}{5}$*d.*

XXII. FRUSTUM OF A CONE OR PYRAMID.

(1) $383\frac{3}{7}$ cub. ft., or 383 cub. ft. $740\frac{4}{7}$ cub. in.
(2) $2325\frac{5}{7}$ cub. ft., or 2325 cub. ft. $1234\frac{2}{7}$ cub. in.
(3) 2926 cub. ft.
(4) $73\frac{140}{189}$ cub. ft., or 73 cub. ft. 1280 cub. in.
(5) $556\frac{101}{216}$ cub. ft., or 556 cub. ft. 808 cub. in.
(6) 484 sq. ft. (7) 1980 sq. ft.
(8) $1068\frac{4}{7}$ sq. ft., or 1068 sq. ft. $82\frac{2}{7}$ sq. in.
(9) 1650 sq. ft. (10) 1064 cub. ft.
(11) 4810 cub. ft. (12) 13312 cub. ft.
(13) 430·392 sq. ft. (14) £27 10*s.*
(15) £7 7*s.* $7\frac{3}{4}$*d.* (16) $10\frac{179}{630}$ gallons.
(17) $519\frac{3}{4}$ cub. in. (18) $32\frac{13}{112}$ cub. in.
(19) $180\frac{3}{8}$ cub. ft. (20) $1693\frac{10}{21}$ cub. in.
(21) $99\frac{11}{21}$ cub. ft., or 99 cub. ft. $905\frac{1}{7}$ cub. in.

XXIII. THE WEDGE.

(1) 360 cub. in. (2) 4 cub. ft. 288 cub. in.
(3) 6 cub. ft. 432 cub. in. (4) 1 cub. ft. 312 cub. in.
(5) 2 cub. ft. 324 cub. in. (6) 10 in.
(7) 2 cub. ft. 1152 cub. in. (8) 178·5 cub. ft.
(9) 15 cub. ft. 576 cub. in. (10) 1 cub. ft. 432 cub. in.
(11) 50 sq. ft. 108 sq. in.

XXIV. THE PRISMOID.

(1) 11 cub. ft. 592 cub. in.
(2) 95872 cub. ft.
(3) 5568 cub. ft.
(4) 312320 cub. ft.
(5) $67\frac{5}{9}$ cub. ft., or 67 cub. ft. 960 cub. in.
(6) $34266\frac{2}{3}$ cub. ft.
(7) 188750 cub. ft.
(8) £842 4*s.* $5\frac{1}{3}$*d.*
(9) 530000 cub. ft.
(10) 820512 cub. ft.

XXV. THE SPHERE.

(1) $1437\frac{1}{3}$ cub. in.
(2) $22\frac{11}{24}$ cub. ft., or 22 cub. ft. 792 cub. in.
(3) $103\frac{631}{648}$ cub. ft., or 103 cub. ft. $1682\frac{2}{3}$ cub. in.
(4) $606\frac{3}{8}$ cub. ft., or 606 cub. ft. 648 cub. in.
(5) $2807\frac{7}{24}$ cub. ft., or 2807 cub. ft. 504 cub. in.
(6) 1663·893 cub. ft.
(7) 38808 cub. ft.
(8) 310464 cub. ft.
(9) 2 sq. ft. $26\frac{2}{7}$ sq. in.
(10) 9 sq. ft. 90 sq. in.
(11) 17 sq. ft. 16 sq. in.
(12) 154 sq. ft.
(13) 55·44 sq. ft.
(14) 346·5 sq. ft.
(15) $255\frac{13}{21}$ cub. in.
(16) $286\frac{11}{21}$ cub. in.
(17) 2·644 cub. ft.
(18) $1019\frac{1}{3}$ cub. in.
(19) 5 ft. 3 in.
(20) 7 ft.
(21) 12 ft. 6 in.
(22) 4·2 yds.
(23) 1 ft. 2 in.
(24) 3 ft. 6 in.
(25) 11 ft. 8 in.
(26) 56 ft.
(27) $66\frac{11}{21}$ cub. in.
(28) $8\frac{1349}{1920}$ gallons.
(29) 18 lbs. $6\frac{9}{14}$ oz.
(30) 7 in.
(31) $2906\frac{2}{11}$ cub. in.
(32) $136\frac{8}{9}$ sq. ft.
(33) 693 sq. ft.
(34) 5·46 in.
(35) $51\frac{1}{3}$ lbs.
(36) £95 14*s.* + £5 10*s.* = £101 4*s.*
(37) $54\frac{27}{64}$ lbs.
(38) $5\frac{163}{315}$ lbs.
(39) $163\frac{7}{9}$ lbs.
(40) $75\frac{3}{7}$ sq. ft.
(41) 549 lbs.
(42) 9·506 in.
(43) 14 in.
(44) 29 lbs. $11\frac{1}{5}$ oz.
(45) $20\frac{628}{831}$ gallons; £7 14*s.*
(46) 10·026 in.
(47) $4\frac{1}{2}$ in.
(48) 2 ft. 6 in.
(49) $760\frac{8}{33}$ bullets.
(50) $1018\frac{2}{7}$ sq. in.
(51) 11·81 in.
(52) $1018\frac{2}{7}$ cwts., or 114048 lbs.
(53) 10·095 in. by Note 7, b; and 10·094 by Note 7, a.
(54) $10\frac{1}{2}$ inches.

XXVI. THE SEGMENT OF A SPHERE.

(1) $662\frac{2}{21}$ cub. ft.
(2) $11766\frac{6}{7}$ cub. in., or 6 cub. ft. $1398\frac{6}{7}$ cub. in.
(3) $54476\frac{4}{21}$ cub. ft.
(4) $3486\frac{10}{21}$ cub. in., or 2 cub. ft. $30\frac{10}{21}$ cub. in.
(5) 330 sq. ft. (6) $20\frac{179}{315}$ pints.
(7) $18\frac{791}{1680}$ gallons.
(8) $1571\frac{3}{7}$ sq. in., or 10 sq. ft. $131\frac{3}{7}$ sq. in.
(9) 30·4 + lbs.

XXVII. THE ZONE OF A SPHERE.

(1) $2413\frac{5}{7}$ cub. in., or 1 cub. ft. $685\frac{5}{7}$ cub. in.
(2) $485\frac{4}{7}$ cub. ft. (3) $234\frac{2}{3}$ cub. ft.
(4) $46677\frac{5}{7}$ sq. in., or 324 sq. ft. $21\frac{5}{7}$ sq. in.
(5) $4798\frac{2}{21}$ cub. in., or 2 cub. ft. $1342\frac{2}{21}$ cub. in.
(6) $16\frac{23417}{45360}$ gallons.

XXVIII. THE CIRCULAR RING.

(1) $333\frac{18}{49}$ cub. in. (2) $177\frac{39}{49}$ cub. in. (3) $125\frac{5}{392}$ cub. in.
(4) $1700\frac{17}{98}$ oz., or 106 lbs. $4\frac{17}{98}$ oz. (5) £2 1*s*. $11\frac{37}{49}$
(6) $998501\frac{43}{49}$ oz., or 27 tons 17 cwts. 22 lbs. $5\frac{43}{49}$ oz.

XXIX. IRREGULAR SOLIDS.

(1) 5 cub. ft. 432 cub. in. (2) 231 cub. in.
(3) 8 cub. ft. 576 cub. in. (4) $627\frac{3}{31}$ cub. in.
(5) $975\frac{15}{31}$ cub. in. (6) $4\frac{1}{2}$ cub. ft.
(7) 1200 cub. in. (8) 300 cub. in.
(9) $15\frac{17}{25}$ cub. ft., or 15 cub. ft. $1175\frac{1}{25}$ cub. in.
(10) $28\frac{84}{125}$ cub. ft., or 28 cub. ft. $1161\frac{27}{125}$ cub. in.
(11) 252 cub. ft. (approximately).

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